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Temperature and damage in continuum models for tumor growth: analytic results and optimal control

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Contents

Abstract

Tumor growth is a complex phenomenon, driven by the interplay of genetic, biochemical, and mechanical factors, and its understanding remains a major challenge in modern medicine. Mathematical oncology offers a powerful quantitative framework to unravel the underlying mechanisms and to pave the way toward personalized treatments. This thesis falls within this context, introducing systems of partial differential equations (PDEs) intended to capture the fundamental biological and physical processes involved in cancer progression. Starting from well-established models that couple tumor dynamics with nutrient diffusion, we incorporate the evolution of additional relevant quantities, thereby enriching both the mathematical structure and the modeling viewpoint.

Once the PDE system is formulated and an appropriate notion of weak solution is introduced, a fundamental question concerns the well-posedness of the associated initial-boundary value problem. The standard technique for proving existence consists in constructing a suitable approximation of the system, possibly combined with a regularization of the nonlinear terms. One then derives a priori estimates that guarantee sufficient compactness, allowing the passage to the limit in the approximation parameter and the recovery of a solution to the original problem. If well-posedness and adequate regularity are established, attention can turn to associated optimal control problems. The aim is not only to predict the course of the disease but also to optimize therapeutic strategies by adjusting the type, timing, and dosage of treatments to achieve the best possible outcome for the patient, meaning maximum reduction of the tumor and minimum drug-related side effects. From a mathematical perspective, this is achieved by introducing control terms in the source functions of the system, representing the therapies. These controls are chosen within an admissible set, ensuring that for each fixed control, the PDE system admits a unique solution. A first step is to prove the existence of an optimal control, namely one that minimizes a suitable cost functional depending on both the control and the corresponding solution of the PDE system. This gives rise to a nonlinear and nonconvex minimization problem subject to PDE constraints, typically addressed via the direct method in the Calculus of Variations. A subsequent major goal is the derivation of first-order necessary optimality conditions, expressed in the form of a variational inequality.

This thesis introduces three mathematical models. The first is a nonisothermal phase field system of Caginalp type, designed to describe tumor progression under thermal therapies. Despite the biological relevance of temperature, this aspect has received limited attention in the literature; the approach proposed here provides a novel perspective and a solid foundation for future developments. Well-posedness for the related initial-boundary value problem and additional regularity are proven. In particular, the existence of a weak solution is demonstrated through a two-step approximation procedure, involving regularization of the potential and a Faedo–Galerkin discretization scheme. The second model builds on a Cahn–Hilliard-type system already incorporating mechanical effects—reflecting the viscoelastic properties of tissues—by introducing surgery-induced damage, a complete novelty in this field. In addition to the usual technical difficulties, such as the lack of mass conservation, this system presents further challenges due to the nonlinear coupling between the equations, in particular between the balance of forces and the differential inclusion governing the damage variable. The latter is by itself highly nonlinear, presenting both a p-Laplacian operator and the subdifferential of a nonsmooth convex potential. The existence of weak solutions is established through a carefully chosen time-discretisation scheme. However, due to the high nonlinearity, uniqueness remains an open problem. Nevertheless, if a suitable non-degenerative operator replaces the p-Laplacian, well-posedness can be obtained, giving new insights for further research. The third model is tailored to brain tumors and integrates lactate metabolism, viscoelasticity, and tissue damage, together with the action of cytotoxic and lactate-targeting drugs. Here, the evolution of the tumor is described through a Fisher–KPP-type equation. The existence of weak solutions is proved by a Schauder fixed-point argument combined with a regularization of the potential. After showing additional regularity and uniqueness, an associated optimal control problem is addressed.

Declarations

The author hereby declares that the present thesis is the result of original research work carried out independently under the supervision of Prof. Pierluigi Colli and Prof. Elisabetta Rocca, except where explicitly stated otherwise.

This thesis is largely based on and inspired by the following published or submitted papers.

- G. Cavalleri. "A phase field model of Cahn–Hilliard type for tumour growth with mechanical effects and damage". In: *J. Math. Anal. Appl.* 550.2 (2025), p. 129627.
- G. Cavalleri, P. Colli, A. Miranville, and E. Rocca. "On a brain tumor growth model with lactate metabolism, viscoelastic effects, and tissue damage". In: *Nonlinear Anal.: Real World Appl.* 87 (2026), p. 104419.
- G. Cavalleri and A. Miranville, "Optimal control on a brain tumor growth model with lactate metabolism, viscoelastic effects, and tissue damage". In: *Discrete Contin. Dyn. Syst. - B* (2025).
- G. Cavalleri, P. Colli, and E. Rocca. "Well-posedness for a fourth-order nonisothermal tumor growth model of Caginalp type". Preprint: arXiv:2508.07979 [math.AP], (2025), pp. 1-35. Submitted.

At the beginning of each chapter, the contributions of the aforementioned works to the content of the thesis are explicitly acknowledged and clearly indicated.

Moreover, during the Ph.D., the author has also been involved in the following ongoing project:

- G. Cavalleri, P. Colli, and E. Rocca. "Optimal control on a fourth-order nonisothermal tumor growth model of Caginalp type".

The author further declares that there is no conflict of interest. This manuscript has not been submitted, in whole or in part, for other purposes at any other university.

Chapter 1

Introduction

Despite significant advances in modern medicine, cancer remains one of the leading causes of death worldwide. According to global estimates from the International Agency for Research on Cancer (IARC), in 2022, there were nearly 20 million new cancer cases and 9.7 million cancer-related deaths, accounting for almost one in six deaths (see [Bra+24; Fer+21]). Moreover, due to the actual rates in population growth and aging, these numbers are set to rise: demographics-based predictions indicate that the number of new cases of cancer will reach 35 million by 2050, a 77% increase with respect to 2022, further intensifying the global burden of the disease (see [Bra+24]). Cancer is an inherently complex multiscale phenomenon in which genetic, biochemical, and mechanical processes act simultaneously. This complexity makes it challenging to distinguish the key drivers of tumor progression from secondary or negligible effects. In addition, tumors exhibit remarkable heterogeneity, while population-based protocols with limited individualization still guide most current therapies. This can result in both undertreatment, which may be fatal, and overtreatment, with the related side effects. For all these reasons, mathematical oncology has come to be recognized as a powerful and indispensable tool in cancer research (see, e.g., [Yan+24; Cor+13; Lip+19; AGL23; Fal+21] and the references cited therein). It offers a rigorous framework for investigating the complex interplay among biological processes and therapeutic interventions, and provides quantitative methodologies that can significantly improve both diagnostic precision and prognostic accuracy. One of its primary objectives is to enhance the understanding of the fundamental mechanisms driving tumor development, which are often unclear, and to predict the disease's course accurately. A second major aim of mathematical oncology is to contribute to personalized treatment planning by integrating patient-specific data, possibly from medical imaging, into the design of a therapy optimized by dosage and scheduling. Moreover, it could give valuable insight into the potential efficacy of combinations of therapeutic strategies, which have not been investigated through clinical trials. To achieve these goals, the first fundamental step is the derivation of a mathematical model based on phenomenological observations and physical principles, and the formulation of a related optimal control problem. It is particularly crucial to establish that the model is well-posed, i.e., it admits a unique solution which depends continuously on the given data. Similarly, one must ensure that the associated control problem admits an optimal solution, possibly characterized by suitable necessary conditions. The latter are useful to practically determine the optimal control through a gradient descent method. The subsequent steps concern the numerical implementation of the model, its calibration through patient-specific parameters, and the treatment optimization, which results in a better-informed clinician's decision. This thesis focuses on the analytical aspects described above, without addressing the computational ones.

1.1 Biological framework

The term *cancer* refers to a broad class of diseases characterized by uncontrolled cell proliferation. It can start in any part of the body as a consequence of irreparable DNA damage that disrupts the normal mechanisms controlling cell division and death. In its early stages, a (solid) tumor consists of a small cluster of heterogeneous cells with the abnormal ability to evade *apoptosis* (programmed cell death), ignore growth-inhibitory signals from the surrounding tissue, and proliferate more rapidly than the healthy cells [HW00; HW11]. During the initial *avascular phase*, it draws nutrients (such as oxygen and glucose) diffused within the surrounding tissue by the pre-existing vasculature [Fol76; Fol85; PK89]. To ensure an adequate supply of nutrients, tumor cells can also exploit sophisticated mechanisms. First, tumor cells often overexpress specific membrane transporters, facilitating nutrient uptake [Sca+15; Ish+01; GL16]. This leads to the so-called *active transport* mechanisms, i.e., the preferential flow of nutrients toward tumor cells. Second, tumor cells exhibit *chemotaxis*, i.e., the tendency of tumor cells to migrate toward regions with higher nutrient concentrations [RCP11]. However, as the tumor mass grows beyond a critical size (typically a few millimeters in diameter), the inner regions start to suffer from hypoxia. This results in a characteristic stratification of the tumor, with an outer viable rim of proliferating cells, a quiescent layer of living but non-proliferative cells, and a necrotic core. Moreover, the nutrient deprivation triggers *angiogenesis*, the formation of new blood vessels from the existing vasculature, thereby establishing a direct and sustained supply of nutrients. Finally, tumor cells enter the blood vessels and migrate to other tissues, initiating the process of *metastasis*.

Alongside nutrient availability, several additional mechanisms play a crucial role in influencing tumor growth. Among them, mechanical stress is now well recognized as a factor that can directly affect tumor progression by inhibiting cell proliferation, promoting necrosis, or even altering vascularization patterns (see, e.g., the review [Urc+22]). In particular, [Che+09] proves through in vitro experiments that compressive stress can induce non-hypoxic driven apoptosis, while [Bro+16] shows that mechanically constraining treatments can significantly slow down tumor growth. Conversely, the growth of the tumor itself contributes to the accumulation of stress within the surrounding tissue, creating a nonlinear feedback.

Different cancers require different combinations of treatments, tailored to the tumor type, location, and stage. For primary solid tumors, surgical resection—either partial or complete—often represents a fundamental step in therapy. However, if tumor cell proliferation resumes after surgery, an intriguing question arises as to whether this regrowth is influenced by the surgical site itself. Several mechanisms may contribute to this phenomenon. First, tissue removal inevitably damages local blood vessels and induces edema, both of which may alter nutrient diffusion. In addition, the operated tissue exhibits mechanical properties that differ from those of intact tissue, including altered elasticity (see, e.g., [Moe+17]). In Chapters 4 and 5, we will introduce mathematical models aimed at capturing the impact of surgery-induced tissue damage on growth dynamics. Most of the time, surgery is combined with radiotherapy or chemotherapy. They use, respectively, high radiation doses and cytotoxic drugs to reduce tumor cell proliferation and induce their death. Many other therapies exist, such as hyperthermia therapy, which we are going to take into account in Chapter 3. Hyperthermia is a treatment approach that has been explored for several decades (see, e.g., [Mal+16] and the references therein). It consists in raising the local, regional, or whole-body temperature above 39°C for 30 to 60 minutes, using techniques such as radio wave, microwave, or focused ultrasound heating, perfusion, and thermal chambers (see, e.g., [Hab+11]). It is sometimes used alone, but most frequently is combined with other primary therapies, such as chemotherapy or radiotherapy. Depending on the temperature, it has different effects.

(i) Moderate hyperthermia (below 42°C) increases tumor perfusion, primarily due to heat-induced vasodilatation. This, in turn, improves the delivery of chemotherapeutic (or immunotherapeutic) drugs.

(ii) High hyperthermia (i.e., temperatures between 42°C and 50°C) exerts direct cytotoxic effects and induces vascular damage. Moreover, it impairs DNA repair mechanisms, thereby potentially enhancing the susceptibility of tumor cells to other treatments such as chemotherapy or radiotherapy.

(iii) Thermal ablation (above 50°C), applied directly in situ, results in irreparable cellular damage and consequent apoptosis and necrosis of tumor tissue.

As already pointed out, depending on the location, cancer presents different features that translate into specific prognosis, response to treatments, and, in the end, life expectancy. While the models discussed in Chapters 3 and 4 are not tailored to a specific tumor type, in Chapter 5 we focus on brain tumors. Among them, the more common are *gliomas*, i.e., neoplasms of glial cells. Despite advances in medical research and treatment strategies, the survival rate for brain tumors remains low (see, e.g., [DC16]), and no significant improvement has been observed recently (see, e.g., [Szo+17]). In particular, *glioblastoma*—an aggressive form of glioma—is characterized by invasive growth and genetic-metabolic abnormalities. Due to the oxygen deprivation, hypoxic tumor cells mainly rely on glycolysis to obtain energy, consuming glucose and producing *lactate* as a byproduct. This is known as the *Warburg effect*. On the other hand, lactate has been proven to be more than a waste product: it can be used as a nutrient by oxygenated tumor cells [GOG05]. This leads to some sort of metabolic symbiosis between the two cell populations, suggesting that a new therapeutic strategy may be a drug targeting lactate (cf. [Son+08], [Tan+21], [Guy+22]).

It is now clear that, given the complexity of tumor biology and the multiple interacting processes involved, many factors must be considered when developing a mathematical model. In practice, modeling requires careful choices and simplifications, focusing on the most relevant mechanisms for the specific questions being addressed, which may change from case to case. In the following, when introducing a model (see Section 1.3), we will explicitly point out the assumptions made and the biological phenomena they are intended to capture.

1.2 Modeling framework

Over the past few decades, the interest in tumor growth models and the related literature has increased substantially (see, e.g., [AM04; Byr+06; BLM08; AP08; OHP10; CL10; Fri23] and references therein). Among the various possible classifications, most of them fall into two broad categories: *discrete* and *continuum models*. Discrete models are set at the microscopic scale, explicitly representing individual cells and their interactions. Many of them integrate the single cells' discrete description (for example, through cellular automata, extended Potts, or random walks) with continuous equations for the chemical gradients (see, e.g., [BLM08, Section 4.2] and the reference cited therein). In contrast, continuum models describe tumor dynamics at the tissue level, making them more appropriate for capturing large-scale tumor behavior and long-term evolution. Macroscopic models are often formulated by means of nonlinear partial differential equations (PDEs), which rule the evolution of some locally averaged quantities, such as the tumor cell's density or the concentration of a relevant chemical species. In this thesis, we focus on three continuum models of tumor growth, each tailored to capture specific biological and physical features of the disease. The first two are phase field models, formulated through variants of the Cahn–Hilliard equation, which are well-suited to describe interface dynamics and phase separation phenomena in tumor tissues. The third model is based on a reaction-diffusion equation of Fisher–KPP type, and is specifically designed to describe the invasive behavior of gliomas in brain tissue.

1.2.1 Phase field models of Cahn–Hilliard type

At first glance, the most intuitive way to model a solid tumor is as a cohesive mass of malignant cells embedded within healthy tissue, separated by a *sharp interface* (see Figure 1.1). Translating this notion into mathematical terms naturally leads to free boundary problems, where the tumor is represented as an evolving domain to be determined. Starting from the earliest contributions (see [Gre72; MM78; Ada87]), which incorporated a multilayered structure of the tumor—composed of viable, quiescent, and necrotic cells—together with the presence of growth factors, further extensions have progressively included additional biological processes such as apoptosis, immune response, and angiogenesis (see, e.g., [Byr+06; BLM08] and the references cited therein). Despite their relevance, these models face some limitations. First of all, highly aggressive cancers such as gliomas present an infiltration zone with a low density of isolated tumor cells (see

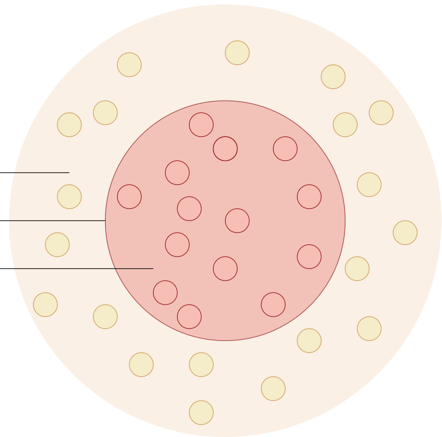

Figure 1.1. Sharp interface model

[SRA08]), making the sharp interface assumption inadequate. Moreover, free boundary problems cannot deal with topological changes in the tumor, such as coalescence or breaking-up phenomena, which typically occur both at the early stages of the proliferation (when the tumor is morphologically unstable, see [CLN03]) and at more advanced stages (when it undergoes metastasis). Finally, without imposing simplifying assumptions, such as radial growth, dealing with free boundary problems may be analytically and numerically challenging. These issues can be overcome by employing a *diffuse interface model*, in which the sharp interface is replaced by a thin layer with both tumor and healthy cells, where a smooth transition takes place (see Figure 1.2). A scalar function

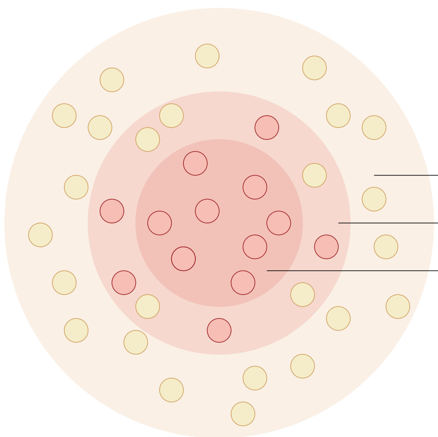

Figure 1.2. Diffuse interface model

φ, called *order parameter*, is introduced, representing the difference in volume fraction of tumor and healthy cells. Roughly speaking, φ is the local relative proportion between the two types of cells, with $\{\varphi = 1\}$ being the tumor tissue, $\{\varphi = -1\}$ the healthy tissue, and $\{-1 < \varphi < 1\}$ the diffuse interface. Notice that here we are implicitly neglecting quiescent and necrotic cells, and the tumor is represented as a single cell population. The evolution of φ is ruled by a suitable modification of the *Cahn–Hilliard equation*. The latter was first introduced by Cahn and Hilliard in 1958 [CH58] to give a continuous description of the *phase separation* process in binary mixtures, modeling spinodal decomposition and coarsening phenomena under the assumptions of isotropy and constant temperature. Since then, it has been widely applied in many different contexts that exhibit pattern formation, ranging from materials science to fluid dynamics, and more recently to mathematical biology. In the following, we introduce it briefly, while referring the reader to [Mir19; Wu22] for a comprehensive overview.

The classical Cahn–Hilliard equation. Suppose that a two-component mixture occupies a domain $\Omega \subseteq \mathbb{R}^d$ with $d = 2, 3$, having boundary $\Gamma := \partial\Omega$ and outward unit normal $\boldsymbol{\nu}$. The classical Cahn–Hilliard equation is a semilinear fourth-order parabolic equation given by

$$\partial_t \varphi + \epsilon \Delta^2 \varphi - \frac{1}{\epsilon} \Delta \Psi'(\varphi) = 0$$

in the parabolic cylinder $\Omega \times (0, T)$, where $T > 0$ is the final time. It can be equivalently rewritten as the following system

$$\begin{aligned} &\partial_t \varphi - \Delta \mu = 0, \\ &\mu = -\epsilon \Delta \varphi + \frac{1}{\epsilon} \Psi'(\varphi), \end{aligned} \tag{1.2.1}$$

having introduced the auxiliary variable μ, called *chemical potential*. The small constant $\epsilon > 0$ is a physical parameter related to the thickness of the interface. The function Ψ is a double-well potential. A thermodynamically relevant example is the logarithmic potential

$$\Psi_{\log}(r) := \frac{\theta_0}{2} \left[(1 - r) \ln(1 - r) + (1 + r) \ln(1 + r) \right] - \frac{\theta_c}{2} r^2, \quad r \in (-1, 1) \tag{1.2.2}$$

where $0 < \theta_0 < \theta_c$ are two constants respectively proportional to the absolute temperature of the system and the critical temperature of phase separation. It is often referred to as the singular potential, since its derivative $\Psi'_{\log}$ presents two vertical asymptotes at $r = \pm 1$. Such a choice guarantees that the order parameter stays within the interval $(-1, 1)$. However, it entails certain technical difficulties. For this reason, it is common to replace $\Psi_{\log}$ with its polynomial approximation, obtained through the fourth-order Taylor expansion around $r = 0$. For a suitable choice of θ_0 and θ_c, and after a translation, this yields

$$\Psi_{\text{reg}}(r) := \frac{1}{4} (1 - r^2)^2, \quad r \in \mathbb{R} \tag{1.2.3}$$

which is smooth and easier to handle both analytically and numerically (see Figure 1.3 for a qualitative comparison). The drawback is that, with this regular potential, the order parameter can no longer be expected to remain within the physically relevant interval $[-1, 1]$, since the Cahn–Hilliard equation does not satisfy a maximum principle. Moreover, notice that the minima, which correspond to the pure phases, are shifted.

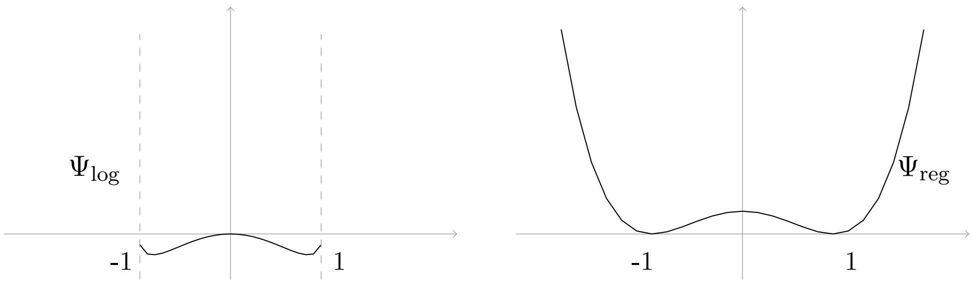

Figure 1.3. The singular potential for $\theta_0 = 2$, $\theta_c = 3$ and the regular potential

Derivation from the mass balance law. The Cahn–Hilliard equation can be phenomenologically derived as follows. We assume that the total free energy of the system is

$$\mathcal{E}(\varphi) := \int_\Omega \left[\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} \Psi(\varphi) \right] \mathrm{d}x,$$

called *Ginzburg–Landau free energy*. The gradient term accounts for heterogeneity of the mixture, modeling the surface energy of the interface. It penalizes sudden spatial changes of the order parameter, preventing the formation of sharp interfaces. The double-well potential reflects the thermodynamic preference of the system for the pure phases (e.g., the two minima) over the diffuse interface, corresponding to intermediate values. The phase separation and the related pattern formation are driven by the competition between $|\nabla \varphi|^2$, smoothing the interfaces, and $\Psi(\varphi)$, favoring pure phases. In the phase separation, the total mass of the system is conserved. Thus, we can write the continuity equation

$$\partial_t \varphi + \operatorname{div} \mathbf{J}_\varphi = 0, \tag{1.2.4}$$

where $\mathbf{J}_\varphi$ is a mass flux to be chosen so that, according to the basic law of thermodynamics, the free energy does not increase in time. Notice that, having introduced $\mathcal{E}$, the chemical potential μ can now be defined as its variational derivative with respect to φ, i.e.,

$$\mu := \frac{\delta \mathcal{E}(\varphi)}{\delta \varphi} = -\epsilon \Delta \varphi + \frac{1}{\epsilon} \Psi'(\varphi), \tag{1.2.5}$$

having assumed homogeneous Neumann boundary conditions

$$\partial_{\boldsymbol{\nu}} \varphi = 0 \tag{1.2.6}$$

on Γ. From (1.2.5), μ can be interpreted as a forcing term proportional to the local distance from equilibrium (see [NS84]). Thus, it is reasonable to assume that the diffusion is driven by the gradient of the chemical potential, leading to the non-Fickian law:

$$\mathbf{J}_\varphi := -\nabla \mu. \tag{1.2.7}$$

Assuming no mass flux at the boundary, the equations hereby obtained are complemented by the following homogeneous Neumann boundary condition for μ

$$\partial_{\boldsymbol{\nu}} \mu = 0 \tag{1.2.8}$$

on Γ. We close the system by adding the initial condition

$$\varphi(0) = \varphi_0. \tag{1.2.9}$$

To conclude, let us briefly comment on the consequences of the choices we have made.

- The boundary condition (1.2.6) yields that interface is orthogonal to Γ.
- By the boundary condition (1.2.8), the total mass, i.e., the integral of the order parameter over the domain, is conserved. Proceeding formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi \,\mathrm{d}x = \int_\Omega \partial_t\varphi \,\mathrm{d}x = \int_\Omega \mathrm{div}(\nabla\mu)\,\mathrm{d}x = \int_\Gamma \nabla\mu\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1} = 0,$$

therefore

$$\int_\Omega \varphi(x,t)\,\mathrm{d}x = \int_\Omega \varphi_0(x)\,\mathrm{d}x, \quad \forall t\in[0,T].$$

- Owing to the boundary condition (1.2.6) and to the choice of the mass flux (1.2.7), if φ is a solution of the Cahn–Hilliard equation, then $\mathcal{E}(\varphi)$ is non-increasing according to the laws of thermodynamics. Again, proceeding formally, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi) &= \int_\Omega \left[\epsilon\nabla\varphi\cdot\nabla(\partial_t\varphi) + \frac{1}{\epsilon}\Psi'(\varphi)\partial_t\varphi\right]\mathrm{d}x\\ &= \int_\Omega \mu\partial_t\varphi\,\mathrm{d}x = \int_\Omega \mu\,\mathrm{div}(\nabla\mu)\,\mathrm{d}x = -\int_\Omega |\nabla\mu|^2\,\mathrm{d}x \le 0.\end{aligned}$$

The viscous Cahn–Hilliard equation. The viscous Cahn–Hilliard system

$$\begin{aligned}&\partial_t\varphi - \Delta\mu = 0,\\ &\mu = \tau\partial_t\varphi - \epsilon\Delta\varphi + \frac{1}{\epsilon}\Psi'(\varphi),\end{aligned} \tag{1.2.10}$$

was first introduced in [Nov88] to account for viscosity effects resulting from phase separation. It can be seen as a regularization of (1.2.1). The constant $\tau > 0$ is a viscosity parameter. Notice that here the chemical potential is simply the sum of the viscous contribution and of the variational derivative of the free energy, i.e.,

$$\mu \coloneqq \tau\partial_t\varphi + \frac{\delta\mathcal{E}}{\delta\varphi}(\varphi). \tag{1.2.11}$$

Notice that $\tau\partial_t\varphi$ adds dissipation to the system. Assuming again homogeneous Neumann boundary conditions, this can be seen from the associated energy identity, which takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi) = -\int_\Omega \left[|\nabla\mu|^2 + \tau|\partial_t\varphi|^2\right]\mathrm{d}x, \tag{1.2.12}$$

where the right-hand side is obviously non-positive. Notice that the mass conservation still holds in this case.

Adapting the Cahn–Hilliard equation to tumor modeling. The classical Cahn–Hilliard equation is derived under the hypothesis of mass conservation, which is clearly not satisfied when it comes to tumor growth. Proliferation from nutrient consumption, apoptosis, and the effects of cytotoxic drugs can be incorporated by introducing a phenomenologically chosen mass source U on the right-hand side of (1.2.4), and the same modification applies to its viscous variant. As we will see in Chapters 3 and 4, the absence of mass conservation is related to some technical challenges, and the source—typically nonlinear—has to be selected carefully in order to guarantee the existence of the solutions. Moreover, the Cahn–Hilliard-type equation—describing the evolution of the tumor—is often coupled with additional equations accounting for biologically relevant quantities. The diffusion of nutrients is commonly taken into account through a reaction-diffusion equation, but various works also incorporate the evolution of relevant biomarkers, more phases, elastic effects, and fluid flows (see, e.g., [FGR15; Gar+16; GL17; Fri+18; Gar+18; LW18; EGN21; Col+20; GLS21a; GKT22; KS22] and the references therein). These contributions enrich the free energy functional by adding further terms beyond the classical Ginzburg–Landau one. Consequently, they naturally appear in the Cahn–Hilliard dynamics through the chemical potential, since it is defined as the variational derivative of the free energy.

1.2.2 The inclusion of thermal effects: the conserved Caginalp system

As previously observed, the Cahn–Hilliard equation and its viscous variant are formulated assuming constant temperature, which is not always reasonable from the modeling point of view. To overcome this limitation, several nonisothermal phase transition models have been proposed (see, e.g., [Cag88; Cag90; PF90; AP92a; AP92b] and the more recent [MS05; MS21; De +24]). Among them, the conserved Caginalp model has been discussed and analyzed over the years (see, e.g., [Cag88; Cag90; GPS07; Mir13; Col+23; Col+24] and references therein). The PDE system is basically obtained by coupling the possibly viscous Cahn–Hilliard equation with the heat equation. Its simpler formulation takes the form

$$\begin{aligned} &\partial_t(\theta + \ell\varphi) - \Delta\theta = u, \\ &\partial_t\varphi - \Delta\mu = 0, \\ &\mu = \tau\partial_t\varphi - \epsilon\Delta\varphi + \frac{1}{\epsilon}\Psi'(\varphi) - \Lambda\theta, \end{aligned} \tag{1.2.13}$$

where θ is the (relative) temperature with respect to a certain critical value normalized to 0, u is a heat source, ℓ and Λ are positive constants related to the tissue's latent heat, and $\tau \geq 0$ is a viscous coefficient. It is referred to as the conserved Caginalp model (in contrast to the nonconserved variant, see [Cag86]) because the total mass of the order parameter remains conserved under homogeneous Neumann boundary conditions. The PDE system (1.2.13) can be derived in several ways. We do it following the approach from Caginalp (see [Cag88; Cag90] but also [BS96, Chapter 4, Example 4.4.2]). The evolution of temperature is governed by the heat balance

$$\partial_t(\theta + \ell\varphi) - \Delta\theta = u,$$

coupled with the mass balance

$$\partial_t \varphi + \mathrm{div}(-\nabla \mu) = 0, \tag{1.2.14}$$

where μ is the chemical potential of the system, defined as in (1.2.11) following [Nov88]. Finally, the total free energy functional is given by

$$\mathcal{E}(\theta, \varphi) = \int_\Omega \left[\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} \Psi(\varphi) - \Lambda \theta \varphi \right] \,\mathrm{d}x, \tag{1.2.15}$$

accounting for the temperature through the last coupling term, which may be understood as the part of the free energy corresponding to temperature times the entropy of the system (see [Cag86] but also [BS96]). From the thermodynamic consistency viewpoint, notice that, even assuming $u = 0$, the energy dissipation is not guaranteed. In fact, proceeding as before, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\theta, \varphi) = - \int_\Omega \left[|\nabla \mu|^2 + \tau |\partial_t \varphi|^2 \right] \,\mathrm{d}x - \Lambda \int_\Omega \partial_t \theta \varphi \,\mathrm{d}x,$$

where the last addend does not have a definite sign in general, so $\mathcal{E}(\theta, \varphi)$ may either increase or decrease depending on the evolution of θ and φ (see, e.g., [BS96, Chapter 4, Example 4.4.2] for more details).

1.2.3 Reaction diffusion Fisher–KPP models

Reaction-diffusion models are widely used in mathematical oncology to describe tumor growth. They are relatively simple compared to other classes of equations, yet have proven to give valuable insight. Let φ denote the tumor cell density. Writing the continuum equation for mass conservation, and assuming that the diffusion is gradient-driven according to Fick's law, evolution of φ is ruled by the semilinear parabolic equation:

$$\partial_t \varphi - \Delta \varphi = U(\varphi). \tag{1.2.16}$$

The source term U models the tumor cell proliferation. A common choice is the *logistic function*

$$U(\varphi) := p\varphi \left(1 - \frac{\varphi}{N} \right), \tag{1.2.17}$$

where $p > 0$ is a proliferation rate and $N > 0$ is the carrying capacity of the tissue, i.e., the maximum number of cells that can fit per unit volume. Assuming (1.2.17), (1.2.16) is known as the *Fisher–Kolmogorov–Petrovsky–Piskunov* (or *Fisher–KPP* in short) *equation*. It was originally introduced in 1937 by Fisher [Fis37] and independently by Kolmogorov, Petrovsky, and Piskunov [KPP37] to describe population dynamics. Since then, the Fisher–KPP equation has been applied to many fields (see [Mur02; Mur03] for a detailed overview), including brain tumor growth (see, e.g., [SAM00; Mur03; Roc+08; PH13; Rut+17; Hor+21; Mal+25] and the references cited therein) to capture biological invasion. As for the Cahn–Hilliard equation, (1.2.16) is sometimes coupled with other PDEs governing the dynamics of relevant quantities.

1.2.4 The inclusion of damage (and viscoelastic) effects

One of the main contributions of this thesis is the introduction of the concept of tissue damage, which will be incorporated into the models presented in Chapters 4 and 5. In the biological setting considered here, surgery induces a collection of lesions and microlesions that alter nutrient diffusion and the mechanical properties of the affected region, thereby influencing the macroscopic evolution of the tumor. Although this application has not been previously explored in the literature, damage theory has been extensively employed in engineering and materials science to describe the evolution of complex materials (such as concrete or solder alloys), often taking into account elastic or viscoelastic deformations (see, e.g., [BS04; BSS05; BBR08; HK11; RR14; HR15; MRZ10; TM10] and the references chited therein). In this context, damage arises from the breaking of atomic links, leading to the formation of microcracks and microvoids. As a result, the structural integrity of the material and its stiffness are reduced. Within the framework of continuum mechanics, we aim to describe at the macroscopic level the effects of these microscopic movements, following the approach proposed by Frémond and Nedjar [FN95; FN96; Fré02] (see also [Gur96] for a derivation including the Cahn–Hilliard equation). To simplify the exposition and to avoid distinguishing between the Cahn–Hilliard and Fisher–KPP-type models, we temporarily neglect the tumor variable φ, which will be reintroduced in the following Section 1.3. We denote by z an internal scalar variable which models the degree of damage at each material point and takes values in the interval $[0,1]$. At a given time $t \in (0,T)$, $z(x,t) = 1$ indicates that the material at point $x \in \Omega$ is undamaged, $z(x,t) = 0$ corresponds to complete damage, and intermediate values represent partial damage. Alongside the damage, we want to describe the macroscopic deformations of the body Ω. Thus, for every t in the time interval $(0,T)$, we introduce the *deformation field* of each point with respect to the *reference configuration* Ω

$$\begin{aligned} \boldsymbol{y}(\cdot,t) : \Omega &\to \mathbb{R}^d \\ x &\mapsto \boldsymbol{y}(x,t), \end{aligned}$$

and the *displacement field*

$$\begin{aligned} \boldsymbol{u}(\cdot,t) : \Omega &\to \mathbb{R}^d \\ x &\mapsto \boldsymbol{u}(x) \coloneqq \boldsymbol{y}(x,t) - x. \end{aligned}$$

The evolution equations ruling the dynamics of z and $\boldsymbol{u}$ are obtained from the *principle of virtual power*—which is equivalent to the momentum balance laws (see [Fré02; Ant95])—combined with some constitutive assumption involving the system's free energy $\mathcal{E}$ and *pseudopotential of dissipation* $\mathcal{P}$, ensuring consistency with the classical principles of thermodynamics. Before presenting $\mathcal{E}$ and $\mathcal{P}$, we summarize below both the previously introduced and the additional modeling hypotheses adopted here and in Chapters 4 and 5.

- We take the internal constraint

$$0 \le z \le 1$$

to be a physical property of the state variable z. Therefore, according to the literature [FN95], it must be imposed through the free energy $\mathcal{E}$ which describes properties of the state variables (in contrast to $\mathcal{P}$, describing properties related to their time derivatives).

- Within our biological setting, the damage process is *reversible*, which means that the tissue can recover. In many damage models, the process is instead treated as irreversible, typically enforced via the constraint

$$\partial_t z \leq 0$$

  imposed thought the pseudopotential of dissipation $\mathcal{P}$ (see, e.g., [HK11; RR14; Hei+17]).

- We assume that Ω, the region of the body where cancer is developing, is made of a *viscoelastic material*, i.e., a medium combining features of both elastic solids and viscous fluids. Elastic solids are characterized by their ability to return instantaneously to the original, undeformed configuration once the applied load is removed. They store mechanical energy in the form of elastic energy without dissipation, and the corresponding stress (i.e., the internal force per unit area acting within the body) depends exclusively on the current state of deformation. In contrast, Newtonian viscous fluids dissipate all the mechanical energy supplied through deformation and exhibit no recovery of their initial configuration once the load is removed; for such materials, the stress depends solely on the rate of deformation. The overall response of viscoelastic materials is therefore time-dependent, reflecting both energy storage and dissipation.

- We require that the displacement gradient $\nabla \boldsymbol{u}$ is small. This way, the *Green–St. Venant strain tensor*—which measures the deviation between the given deformation and a rigid one—can be approximated with the *linearized strain tensor*

$$\varepsilon(\boldsymbol{u}) := \frac{(\nabla \boldsymbol{u})^t + \nabla \boldsymbol{u}}{2},$$

  i.e., the symmetric gradient of $\boldsymbol{u}$ (see, e.g., [Cia88]).

Explicitly, the free energy of our system is defined as

$$\mathcal{E}(\boldsymbol{u}, z) := \int_\Omega E(\varepsilon(\boldsymbol{u}), z, \nabla z)\, \mathrm{d}x,$$

with the free energy density given by

$$E(\varepsilon(\boldsymbol{u}), z, \nabla z) := \frac{1}{p}|\nabla z|^p + \widehat{\beta}(z) + \widehat{\pi}(z) + W(\varepsilon(\boldsymbol{u}), z).$$

According to the gradient theory in damage processes, the gradient term models the influence of damage at a material point on the surrounding undamaged material. The non-smooth convex $\widehat{\beta}$ allows us to impose physical constraints on the variable z. A simple and classical example to keep in mind is the indicator function of the interval $[0,1]$, i.e.,

$$\widehat{\beta}(r) = I_{[0,1]}(r) = \begin{cases} 0 & \text{if } r \in [0,1], \\ +\infty & \text{otherwise,} \end{cases}$$

see Figure 1.4. In particular, it would ensure that the damage z takes values in the physically significant range $[0,1]$. The function $\widehat{\pi}$ is a smooth and, in general, concave perturbation with at most quadratic growth (see Figure 1.4 for a simple choice). The

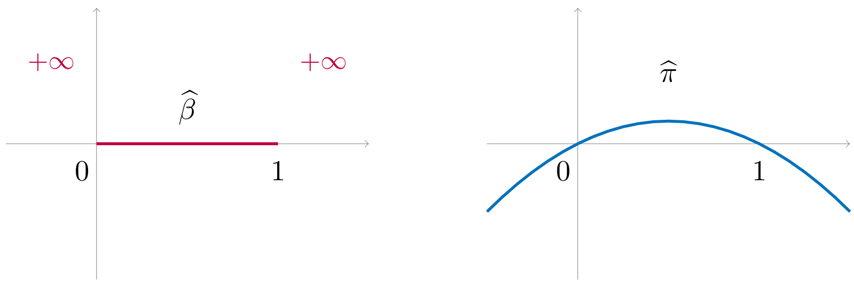

Figure 1.4. $\widehat{\beta}(r) = I_{[0,1]}(r)$ and $\widehat{\pi}(r) = \frac{r(1-r)}{2}$

last term W is the elastic energy density of the system. A common ansatz is to assume that W has a quadratic dependence on the strain tensor, namely

$$W(\varepsilon(\boldsymbol{u}), z) := \frac{1}{2}\mathcal{C}(z)\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}),$$

where $\mathcal{C}$ denotes a fourth-order elastic tensor, which may depend both on the spatial point in the reference configuration Ω and on the damage variable z. Here, the symbol $:$ stands for the Frobenius inner product between matrices. Finally, we define the pseudopotential of dissipation

$$\mathcal{P}(\varepsilon(\partial_t \boldsymbol{u}), \partial_t z) := \int_\Omega P(\varepsilon(\partial_t \boldsymbol{u}), \partial_t z) \,\mathrm{d}x,$$

where the density reads

$$P(\varepsilon(\partial_t \boldsymbol{u}), \partial_t z) := \frac{1}{2}\mathcal{V}(z)\varepsilon(\partial_t \boldsymbol{u}) : \varepsilon(\partial_t \boldsymbol{u}) + \frac{1}{2}|\partial_t z|^2.$$

It depends on the damage time derivative $\partial_t z$ and the macroscopic symmetric strain rate $\varepsilon(\partial_t \boldsymbol{u})$, which are the dissipative variables of our problem. The fourth-order viscous tensor term $\mathcal{V}$ accounts for the friction between neighboring cells moving at different velocities. Although it may depend on the damage variable z—as well as on the spatial point in the reference configuration Ω—for brevity, we denote it by P rather than P_z.

Displacement equation. The application of the principle of virtual power to $\boldsymbol{u}$ leads to the classical *balance of linear momentum*

$$\kappa \partial_{tt} \boldsymbol{u} - \operatorname{div} \mathcal{T} = \boldsymbol{f}. \tag{1.2.18}$$

If $\kappa \neq 0$, inertial effects are taken into account; otherwise, we are considering a quasi-static approximation. The term $\boldsymbol{f}$ is an external load, while $\mathcal{T} = (\mathcal{T}_{ij})$ is the *Piola–Kirchhoff stress tensor*—$\mathcal{T}_{ij}$ is the ith component of the force per unit area in the reference configuration, measured across a surface element with unit normal given by the jth standard Euclidean basis vector. Since we consider a viscoelastic framework, the simplest choice is to postulate the behavior of a *Kelvin–Voigt material*, in which the stress is the sum of a purely elastic nondissipative part and a purely viscous dissipative part:

$$\mathcal{T} = \mathcal{T}^{\mathrm{nd}} + \mathcal{T}^{\mathrm{d}}.$$

Assuming a hyperelastic and "hyperviscous" stress/strain response, we use the ansatz

$$\mathcal{T}^{\mathrm{nd}} := \partial_{\varepsilon(\boldsymbol{u})} E = W_{,\varepsilon}(\varepsilon(\boldsymbol{u}), z) = \mathcal{C}(z)\varepsilon(\boldsymbol{u}), \qquad \mathcal{T}^{\mathrm{d}} := \partial_{\varepsilon(\partial_t \boldsymbol{u})} P = \mathcal{V}(z)\varepsilon(\partial_t \boldsymbol{u}),$$

Here we adopt the standard notation according to which $W_{,\varepsilon}$ is the derivative of W with respect to its first entry $\varepsilon(\boldsymbol{u})$ and the same for the other variable. For more details, the interested reader may also refer to [MR15, Section 4.1], [MH94, Sections 0.1–0.3].

Damage equation. The principle of virtual power with respect to z yields the micro-force balance law

$$B - \operatorname{div} \mathbf{H} = 0. \tag{1.2.19}$$

The quantity B represents the internal microforces and, again following Frémond's approach [Fré02], we postulate that it can be additively decomposed into a dissipative and a non-dissipative part

$$B = B^{\mathrm{nd}} + B^{\mathrm{d}} \qquad \text{with} \qquad \begin{aligned} B^{\mathrm{nd}} &\in \partial_z E = \partial\widehat{\beta}(z) + \widehat{\pi}'(z) + W_{,z}(\varepsilon(\boldsymbol{u}), z), \\ B^{\mathrm{d}} &= \partial_{\partial_t z} P = \partial_t z. \end{aligned}$$

Without entering into the mathematical details, $\partial_z E$ and $\partial\widehat{\beta}$ have to be interpreted as subdifferentials in the sense of convex analysis, and this justifies the presence of the inclusion instead of an equality (see Section 2.5). In the following, we will employ the notation $\beta := \partial\widehat{\beta}$ and $\pi := \widehat{\pi}'$ (see Figure 1.5 for a simple example).

The term $\mathbf{H}$ is the internal micro-stress and is defined by

$$\mathbf{H} = \mathbf{H}^{\mathrm{nd}} + \mathbf{H}^{\mathrm{d}} \qquad \text{with} \qquad \begin{aligned} \mathbf{H}^{\mathrm{nd}} &= \partial_{\nabla z} E = |\nabla z|^{p-2}\nabla z, \\ \mathbf{H}^{\mathrm{d}} &= \partial_{\nabla(\partial_t z)} P = 0. \end{aligned}$$

We assume that the sum of the external microforces acting on the body is equal to zero. Notice that here, according to the literature, we are neglecting the acceleration forces of the microscopic motions as well as external microforces.

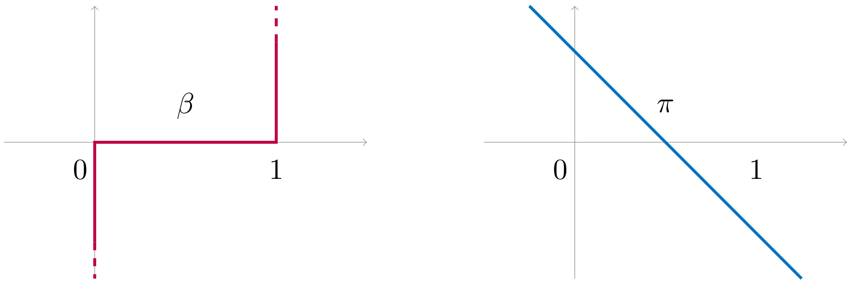

Figure 1.5. $\beta(r) = \partial I_{[0,1]}(r)$ and $\pi(r) = \widehat{\pi}'(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{r(1-r)}{2}\right) = \dfrac{1}{2} - r$

1.3 Mathematical models of tumor growth

In the following, we introduce the three tumor growth models whose study is the objective of this thesis. For each case, we present the underlying biological assumptions and the simplifications that have been made. The presentation closely follows the papers in which these models were originally formulated and analyzed, and the corresponding reference will be explicitly cited. These models, although different in structure and purpose, share a common goal: to provide a reliable and mathematically sound description of tumor evolution that can serve as a foundation for more advanced, patient-specific, and computationally implementable approaches.

1.3.1 A nonisothermal model

In this section, we present a Caginalp-type model for nonisothermal tumor growth, accounting for thermotherapy and the presence of a nutrient. We neglect differentiation among tumor cells and suppose that the tumor is in its avascular phase. The PDE system, introduced and analyzed in [CCR25], reads as

$$\partial_t(\theta + \ell\varphi) - \Delta\theta = u, \tag{1.3.1a}$$

$$\partial_t\varphi - \Delta\mu = U(\theta, \varphi, \sigma), \tag{1.3.1b}$$

$$\mu = \tau\partial_t\varphi - \epsilon\Delta\varphi + \frac{1}{\epsilon}\Psi'(\varphi) - \chi\sigma - \Lambda\theta, \tag{1.3.1c}$$

$$\partial_t\sigma - \Delta(\sigma - \chi\varphi) = S(\theta, \varphi, \sigma), \tag{1.3.1d}$$

in the parabolic cylinder $Q := \Omega \times (0, T)$, coupled with the boundary conditions

$$\partial_{\boldsymbol{\nu}}\theta = \partial_{\boldsymbol{\nu}}\varphi = \partial_{\boldsymbol{\nu}}\mu = \partial_{\boldsymbol{\nu}}\sigma = 0 \tag{1.3.2}$$

on $\Sigma := \Gamma \times (0, T)$, and with the initial conditions

$$\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \tag{1.3.3}$$

in Ω. The subsystem (1.3.1a)–(1.3.1c) is of Caginalp type (cf. (1.2.13)), and is made of the second-order parabolic equation (1.3.1a), ruling the evolution of the relative temperature θ

and the (possibly) viscous Cahn–Hilliard system (1.3.1b)–(1.3.1c), governing the evolution of the phase field variable φ. The parameter ϵ, representing the thickness of the interfacial layer, is set equal to 1 without loss of generality, since it does not affect the mathematical analysis we are interested in. The parabolic equation (1.3.1d) prescribes the dynamics of the nutrient concentration σ that we suppose is consumed only by tumor cells. The main novelty of this model is the fact that it is not isothermal. In fact, among the extensive literature on phase field models for tumor growth (see the previous Section 1.2), many biological quantities are taken into account. However, to the best of our knowledge, only the recent contribution [Ipo22] incorporates temperature effects. In the paper [Ipo22], a thermodynamically consistent nonisothermal diffuse interface model for tumor growth is proposed and analyzed. The system is designed to capture the interplay between temperature variation, nutrient transport, cell proliferation, and apoptosis, and it is rigorously studied through an entropy balance approach, leading to the existence of weak solutions for the corresponding initial-boundary value problem. While our approach shares with [Ipo22] the interest in coupling phase field dynamics with thermal effects and a tumor growth process, we address a different system and adopt a distinct analytical strategy. In particular, beyond proving the existence of weak solutions, we also establish higher regularity and continuous dependence on the data. Moreover, building on these analytical results, an associated optimal control problem is currently under investigation within the ongoing project [CCR].

Derivation of the model. The model (1.3.1)–(1.3.3) is derived following the approach from Caginalp, as in the previous Section 1.2. The only difference is that, in this case, we are also taking into account the presence of the nutrient, which consequently appears in the free energy:

$$\mathcal{E}(\theta,\varphi,\sigma) = \int_\Omega \left[\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi) + \frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi) - \Lambda\theta\varphi\right] \mathrm{d}x. \tag{1.3.4}$$

The term $\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)$ is of classical Ginzburg–Landau type, and the contribution $\frac{1}{2}|\sigma|^2$ reflects the energetic cost associated with the presence of nutrients, implying that high concentrations increase the system's free energy. Notice that we assume that there is no interface between nutrient-rich and nutrient-poor phases, so that the dynamics of the nutrient is simply governed by diffusion (cf. [HZO12]). The two final addenda couple the Cahn–Hilliard equation with the nutrient and with the temperature equations. The nonnegative constants χ and Λ represent, respectively, a transport coefficient (modeling effects such as chemotaxis and active transport) and an already introduced parameter related to the tissue's latent heat. The nutrient equation is also obtained by a mass balance,

$$\partial_t\sigma + \operatorname{div}\mathbf{J}_\sigma = S,$$

where S is a phenomenologically chosen source/sink of nutrients, and the mass flux is

$$\mathbf{J}_\sigma := -\nabla\left(\frac{\delta\mathcal{E}}{\delta\sigma}(\theta,\varphi,\sigma)\right) = -\nabla\sigma + \chi\nabla\varphi.$$

Notice that, thanks to the coupling term $-\chi\sigma\varphi$ in the free energy, the nutrient equation (1.3.1d) contains the term $\operatorname{div}(\chi\nabla\varphi)$, accounting for the active transport. Similarly, writing the mass balance for φ as in (1.2.14), the term $\operatorname{div}(\chi\nabla\sigma)$ appears in the Cahn–Hilliard equation, accounting for chemotaxis (for more details, see the [Gar+16; EGN21]).

Boundary and initial conditions. We couple the system (1.3.1) hereby obtained with the no-flux boundary conditions (1.3.2), which reflect the assumption that the system is effectively isolated from its surroundings, and with the initial conditions (1.3.3).

Choice of the sources. We introduce the mass and nutrient sources that we are going to employ throughout Chapter 3. Starting from U, we assume

$$U(\theta, \varphi, \sigma) := (\lambda_P\sigma - \lambda_A - \lambda_E\theta)\mathbb{h}(\varphi).$$

According to the literature (see [GL17]), we assume the mechanisms controlling cell division to be suppressed in tumor cells. Thus, proliferation is limited only by the availability of nutrients. We model it with the term $\lambda_p\sigma$, where λ_p is a fixed proliferation coefficient. We also suppose that tumor cells only die because of apoptosis, and we denote with λ_a the constant apoptosis rate. Moreover, here we assume that θ has cytotoxic effects proportional to the temperature, which we incorporate through the term $-\lambda_E\theta$, where λ_E is a fixed positive parameter. The function $\mathbb{h}$ guarantees that the phenomena we have just described are proportional to the tumor cells available in a certain area. For example, $\mathbb{h}$ may be a monotone increasing function which is 0 where $\{\varphi = -1\}$ and 1 where $\{\varphi = 1\}$, as in Figure 1.6. On the other hand, we assume the nutrient source S to

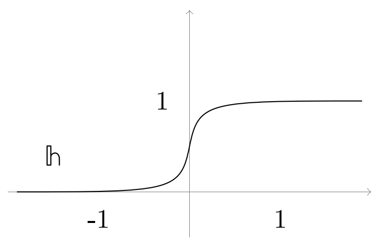

Figure 1.6. Possible choice for the function $\mathbb{h}$

be of the form

$$S(\theta, \varphi, \sigma) := -\lambda_C\sigma\mathbb{h}(\varphi) + \lambda_B(\sigma_B - \sigma) - \lambda_D\sigma\mathbb{k}(\theta).$$

The first two addends are again compliant with [GL17]. The term $-\lambda_C\sigma\mathbb{h}(\varphi)$ models the fact that nutrient consumption is higher where tumor cell density is higher, while $\lambda_B(\sigma_B - \sigma)$ accounts for nutrient supply from the pre-existing capillaries. A novel aspect of the model is the inclusion of the term $-\lambda_D\sigma\mathbb{k}(\theta)$, which describes the influence of temperature on nutrient absorption—e.g., due to vasodilation effects that enhance nutrient transport in warmer tissue regions (see, e.g., [Son+05]).

1.3.2 A model including tissue damage and mechanical effects

In this section, we introduce a phase field model for the evolution of a young tumor, meaning that we again suppose it is in the avascular phase and there is no differentiation between different types of tumor cells. We will take into account the presence of a nutrient, the mechanical behavior of the tissue, modeled as a viscoelastic material, and the local tissue damage caused by surgery. What follows is mainly based on [Cav25]. We are interested in the following PDE system

$$\partial_t\varphi - \Delta\mu = U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z), \tag{1.3.5a}$$

$$\mu = -\epsilon\Delta\varphi + \frac{1}{\epsilon}\Psi'(\varphi) + W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z), \tag{1.3.5b}$$

$$\partial_t\sigma - \Delta\sigma = S(\varphi, \sigma, z), \tag{1.3.5c}$$

$$\kappa\partial_{tt}\boldsymbol{u} - \operatorname{div}\left[a(z)\mathcal{V}\varepsilon(\partial_t\boldsymbol{u}) + W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)\right] = \boldsymbol{0}, \tag{1.3.5d}$$

$$\partial_t z - \Delta_p z + \beta(z) + \pi(z) + W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z) \ni 0, \tag{1.3.5e}$$

posed in Q. The Cahn–Hilliard equation given by the combination of (1.3.5a)–(1.3.5b) describes the evolution of the order parameter φ. The reaction-diffusion equation (1.3.5c) rules the diffusion of σ, that is, the concentration of the nutrient. The hyperbolic equation (1.3.5d) describes the dynamics for the vectorial variable $\boldsymbol{u}$, the small displacement field of each point with respect to the reference undeformed configuration. The fixed and positive parameter κ is supposed to be small and represents the fact that tumor growth occurs at a much larger timescale than the tissue relaxation into mechanical equilibrium. For simplicity and without any loss of generality, later on, we will set $\epsilon = \kappa = 1$. Finally, the differential inclusion (1.3.5e) represents the evolution law for local tissue damage z. While the influence of the nutrient dynamics and of viscoelastic effects on tumor growth is already deeply investigated in the literature (see, e.g., [GLS21a; GKT22; GT24]), the role of the damage is a novelty in this field, first introduced in [Cav25]. However, the impact of the damage and viscoelasticity in phase separation processes has been thoroughly explored in various modeling studies within the field of materials science (see, e.g., [HK11; HK15; Hei+17] and the references cited therein).

Derivation of the model. The evolution of our system is driven by classical thermodynamic principles and relies on a total energy $\mathcal{E}$ and a pseudopotential of dissipation $\mathcal{P}$. The total energy of our system is

$$\mathcal{E}(\varphi, \sigma, \boldsymbol{u}, z) = \int_\Omega E(\varphi, \nabla\varphi, \sigma, \varepsilon(\boldsymbol{u}), \partial_t\boldsymbol{u}, z, \nabla z)\,\mathrm{d}x$$

where the energy density E is the sum of a generalized free energy density and the kinetic energy density. We postulate it has the following form:

$$\begin{aligned} &E(\varphi, \nabla\varphi, \sigma, \varepsilon(\boldsymbol{u}), \partial_t\boldsymbol{u}, z, \nabla z) \\ &\quad = \frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi) + \frac{1}{2}|\sigma|^2 + \frac{1}{2}|\partial_t\boldsymbol{u}|^2 + \frac{1}{p}|\nabla z|^p + \widehat{\beta}(z) + \widehat{\pi}(z) + W(\varphi, \varepsilon(\boldsymbol{u}), z). \end{aligned}$$

The term $\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)$ is the classical contribution of Ginzburg–Landau type and $\frac{1}{2}|\sigma|^2$ results from the presence of the nutrient. The system's kinetic energy is given by $\frac{1}{2}|\partial_t \boldsymbol{u}|^2$. Regarding the damage, $\frac{1}{p}|\nabla z|^p + \widehat{\beta}(z) + \widehat{\pi}(z)$ is an interaction free energy, compliant with the one introduced in Section 1.2.4. Lastly, W is the elastic energy density. The pseudopotential of dissipation is

$$\mathcal{P}(\varepsilon(\partial_t \boldsymbol{u}), \partial_t z) = \int_\Omega P(\varepsilon(\partial_t \boldsymbol{u}), \partial_t z)\,\mathrm{d}x = \int_\Omega \frac{1}{2} a(z)\varepsilon(\partial_t \boldsymbol{u}) : \mathcal{V}\varepsilon(\partial_t \boldsymbol{u}) + \frac{1}{2}|\partial_t z|^2\,\mathrm{d}x,$$

as in the previous Section 1.2.4. Notice that P depends also on the damage z through the viscous coefficient a, but we use the notation P instead of a more precise P_z for brevity. Following Frémond's and Gurtin's approach [Fré02; Gur96], our system can be derived starting from balance laws for the involved quantities and then imposing constitutive assumptions so that the system satisfies the second law of thermodynamics, which, in the case of an isothermal system like ours, is written in the form of an energy dissipation inequality (see, e.g., [Gar+16; Hei+17] and the previous Section 1.3.2). The Cahn–Hilliard equation of system (1.3.5a)–(1.3.5b) is derived from a mass balance law with a certain mass source U, as in Section 1.2. Notice that here, due to the presence of the elastic energy in the total free energy of the system, the chemical potential μ, as usual defined as its variational derivative with respect to φ, presents an additional addend:

$$\mu := \frac{\delta \mathcal{E}}{\delta \varphi} = -\Delta\varphi + \Psi'(\varphi) + W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z).$$

As in the previous Section 1.3.1, the nutrient equation (1.3.5c) is obtained from a mass balance law, choosing the mass flux as the variational derivative of $\mathcal{E}$ with respect to φ, and a mass source S. The equations (1.3.5d)–(1.3.5e) governing the displacement and the damage are derived as in Section 1.2.4, with the only difference that here we are neglecting external forces in the equation for macroscopic movements.

Boundary and initial conditions. We assume the system is isolated from the exterior, so we prescribe no-flux conditions for φ, μ, and z. Regarding σ, we allow a more general Robin condition that may also model the boundary supply of the nutrient. We assume that $\boldsymbol{u}$ is zero at the boundary, as in the situation in which the domain is delimited by a rigid part of the body (e.g., a bone) that prevents displacements. Namely, we couple the previous system (1.3.5) with the following boundary conditions

$$\partial_{\boldsymbol{\nu}}\varphi = \partial_{\boldsymbol{\nu}}\mu = 0, \tag{1.3.6a}$$

$$\partial_{\boldsymbol{\nu}}\sigma + \alpha(\sigma - \sigma_\Gamma) = 0, \tag{1.3.6b}$$

$$\boldsymbol{u} = \boldsymbol{0}, \tag{1.3.6c}$$

$$(|\nabla z|^{p-2}\nabla z)\cdot\boldsymbol{\nu} = 0, \tag{1.3.6d}$$

on Σ. The term σ_Γ is the prescribed concentration of the nutrient at the boundary, and α is a given non-negative constant. Notice that, if $\alpha = 0$, we gain a no-flux condition also for the nutrient. The system is supplemented with the initial conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0) = \boldsymbol{v}_0, \quad z(0) = z_0, \tag{1.3.7}$$

in Ω.

Choice of the sources. The nonlinear source U in equation (1.3.5a) accounts for biological mechanisms related to tumor cells' proliferation and death. Explicitly, we make the following choice

$$U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z) \coloneqq \left(\frac{\lambda_p \sigma}{1 + |W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)|} - \lambda_a + f \right) g(\varphi, z), \tag{1.3.8}$$

referring to [GL17; GLS21a]. As it is common, we assume that the proliferation of tumor cells is only limited by the quantity of nutrients available, and that they die only because of apoptosis. As in the previous section, λ_p and λ_a denote the proliferation and apoptosis coefficients, respectively. Furthermore, we consider the presence of mechanical stress caused by surrounding tissues as a factor that can reduce tumor growth. This is expressed by the fact that, if the elastic stress $W_{,\varepsilon}$ grows in modulus, the proliferation term $\lambda_p \sigma$ reduces. We also allow the presence of a medical treatment, modeled by the prescribed function f, that affects proliferation. The function g guarantees that proliferation and apoptosis occur only in the tumor tissue, as well as the effectiveness of the medical care f. A good modeling choice is a non-negative function that vanishes in $\{\varphi = -1\}$, is equal to 1 where $\{\varphi = 1\}$, and is increasing in the variable φ (see Figure 1.6). We also allow the dependence of g on the damage z.
For the choice of the nutrient source S in equation (1.3.5c), we refer to the aforementioned literature, assuming

$$S(\varphi, \sigma, z) \coloneqq -\lambda_c \sigma g(\varphi, z) + \Lambda_c(z)(\sigma_c - \sigma). \tag{1.3.9}$$

The term $-\lambda_c \sigma g(\varphi, z)$ models the fact that the nutrient consumption is higher where the tumor cell density is higher. Here, λ_c is a fixed consumption rate. The term $\Lambda_c(z)(\sigma_c - \sigma)$ is a supply term that takes into account the nutrients provided by the nearby capillaries. Note that the capillary supply rate Λ_c may depend on the local damage parameter z since damage, in the sense of a lesion caused by a surgical procedure, affects the blood vessels.

Choice of the elastic energy density. According to the classical theory of linear elasticity (see, e.g., [Cia88]) and to the previous literature (see, e.g., [GLS21a; GKT22] and [HK11; RR14; HK15; HR15; Hei+17]), we assume that the elastic energy density has the following expression

$$W(\varphi, \varepsilon(\boldsymbol{u}), z) \coloneqq W(x, \varphi, \varepsilon(\boldsymbol{u}), z) = \frac{1}{2} h(z) \mathcal{C}(x)(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : (\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi). \tag{1.3.10}$$

Notice that, even if W may depend on the space variable x, with a slight abuse of notation, we will omit this dependence in the following. Here, $\mathcal{C}$ is a fourth-order elasticity tensor, possibly depending on the point in the reference configuration Ω, whose mathematical requirements will be specified later on. We include the multiplicative, non-negative, and possibly degenerate function h to add dependence on the damage. From the modeling point of view, $\mathcal{C}$ should also depend on the phase φ because tumor tissue and healthy tissue could have a different elastic response to solicitations. However, we were not able to handle such dependence from the mathematical point of view (see Chapter 4 for a more detailed explanation). Finally, the term $\mathcal{R}\varphi$ is the stress-free strain (also called eigenstrain), which is the reference strain the material would attain if the tissue were uniform and unstressed at a phase configuration φ. In other words, it is the strain due to growth. As it is common, we assume that it satisfies Vegard's law, i.e., it is given by a linear function of φ, where $\mathcal{R} \in \mathbb{R}^{d\times d}$ is a fixed matrix. With such a choice, the partial derivatives of W that appear in the equations of the PDE system are:

$$W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z) = -h(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : \mathcal{R}, \tag{1.3.11}$$

$$W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z) = h(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi), \tag{1.3.12}$$

$$W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z) = \frac{1}{2}h'(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : (\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi). \tag{1.3.13}$$

1.3.3 A brain model including tissue damage and lactate metabolism

In this section, we introduce a model describing the dynamics of a brain tumor, including lactate metabolism, viscoelastic effects of the tissues, as well as their possible damage. The PDE system is as follows:

$$\partial_t\varphi - \Delta\varphi = U(\varphi, \sigma, z, \chi_1), \tag{1.3.14a}$$

$$\partial_t\sigma - \Delta\sigma + K(\varphi, \sigma, z) = \chi_2 S(\varphi, z), \tag{1.3.14b}$$

$$-\operatorname{div}\left[\mathcal{A}(\varphi, z)\varepsilon(\partial_t\boldsymbol{u}) + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{u})\right] = \boldsymbol{f}, \tag{1.3.14c}$$

$$\partial_t z - \Delta z + \beta(z) + \pi(z) \ni \iota - F(\varphi, \varepsilon(\boldsymbol{u})), \tag{1.3.14d}$$

posed in Q. The model was first introduced in [Cav+26], where well-posedness and regularity of the solution of the related initial-boundary value problem were discussed. Then, an associated optimal control problem was studied in [CM26]. The four nonlinearly coupled PDEs describe the evolution of the concentrations of tumor cells φ, the intracellular lactate σ, the "small" displacement $\boldsymbol{u}$, and the damage parameter z. The assigned functions χ_1 and χ_2 are, respectively, the concentration of a cytotoxic drug that inhibits tumor proliferation and a lactate targeting drug.

Only a few works in the literature address analytically the coupling between tumor dynamics and lactate diffusion (see, e.g., [Aub+05], [Clo+09], [Gui+18], [Che+21], [Che+22]), and even fewer consider optimal control aspects (cf. [Che+24]). In particular, the subsystem (1.3.14a)–(1.3.14b) closely resembles the one analyzed in [Che+22]. The equation for σ is based on the derivation done in [Aub+05] and in [Gui+11]. Actually, in the first brain lactate kinetics models in the literature (cf. [Che+22], [Gui+18] and references therein), the authors dealt with the evolution of both capillary and intracellular lactate concentrations. However, since here we aim to include the displacement and

damage evolution in the model, we neglect the capillary lactate concentration in order to simplify.
Models including the effects of the stress (reducing the proliferation of the tumor) were already introduced and studied in [GLS21a], where a Cahn–Hilliard-type dynamics for the tumor concentration was used in place of the Fischer–KPP-type equation (1.3.14a). Finally, a system coupling tumor growth dynamics of Cahn–Hilliard type together with displacement and damage was first analyzed in [Cav25], which is also the object of Chapter 4 of this thesis. The main novelty of system (1.3.14) relies on the fact that here we include mechanics—assuming a viscoelastic behavior of biological tissues—and tissue damage into a model specifically tailored for brain tumors. It is indeed well known that solid stress can affect tumor growth (see e.g. [Urc+22]) and, at the same time, tumor growth increases mechanical stress. It is worth noting that the study of mechanical effects in tumor growth models becomes particularly relevant in the case of brain tumors (gliomas), where the evolution is strongly influenced by the surrounding tissues (see, e.g., [Alf+17] and references therein). Finally, following the ideas developed in [Cav25], we consider, through equation (1.3.14d), the possible damage effects that could occur, for example, in the case of surgery that causes lesions which, in turn, affect the proliferation of tumor cells.

Derivation of the model. The Fischer–KPP-type equation (1.3.14a) already introduced in Section 1.2.4 describes the evolution of tumor cell concentration φ. The reaction-diffusion equation (1.3.14b) represents the evolution of intracellular lactate production σ (see [Aub+05; Gui+11]). The evolution of the small displacement $\boldsymbol{u}$ is ruled by the vectorial quasi-stationary balance law (1.3.14c) (cf. Section 1.2.4 and [GLS21a; GLS21b]). The two tensors $\mathcal{A}$ and $\mathcal{B}$ describe the elastic and viscous effects. They may depend on the tumor and damage variables, as well as on σ in a non-degenerating way (cf. assumptions (5.1.6) later on). For a more detailed discussion of this dependency, we refer to the following Remark 5.6. The term $\boldsymbol{f}$ represents a given volume force. Finally, the evolution of the damage parameter z is ruled by the evolution inclusion (1.3.14d), where the maximal monotone graph β having bounded domain (in $[0,1]$) forces the variable z to assume the physically meaningful values in between 0 and 1, as in the previous Section 1.3.2. The function π denotes a regular, possibly non-monotone function, ι represents an energy threshold for initiation of damage, while F describes the coupling between the damage and the displacement, along with the tumor concentration. Usually, in damage models (cf. Section 1.2.4, Section 1.3.2), the dependence of F on $\varepsilon(\boldsymbol{u})$ is quadratic because it comes from the derivative of the elastic part of the energy with respect to the damage variable z. However, here we cannot handle a quadratic dependence and so we assume F to be Lipschitz continuous with respect to both φ and $\varepsilon(\boldsymbol{u})$ (cf. assumption (A7) later on, cf. also [KS06]).

Initial and boundary conditions. Regarding the boundary conditions, we assume the system is isolated from the exterior, so we prescribe no-flux conditions for φ and z. As for the lactate σ, we allow for more general Robin conditions, with the physical constants set to 1 for simplicity. Since our model is specifically designed for brain tumors, the domain is confined by a rigid boundary, the cranium, which prevents displacement at the boundary. Consequently, it is a natural choice to impose homogeneous Dirichlet boundary conditions for the displacement $\boldsymbol{u}$. Therefore, we couple the previous system with the following boundary conditions:

$$\partial_{\boldsymbol{\nu}}\varphi = \partial_{\boldsymbol{\nu}} z = 0, \tag{1.3.15a}$$

$$\partial_{\boldsymbol{\nu}}\sigma = \sigma_\Gamma - \sigma, \tag{1.3.15b}$$

$$\boldsymbol{u} = \mathbf{0} \tag{1.3.15c}$$

on Σ. Finally, we consider the following usual initial conditions:

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad z(0) = z_0 \tag{1.3.16}$$

in Ω.

Choice of the sources. Regarding the tumor equation (1.3.14a), we assume the source term U to have of the following expression

$$U(\varphi, \sigma, z, \chi_1) \coloneqq (p(\sigma, z) - \chi_1)\varphi\left(1 - \frac{\varphi}{N}\right) - \varphi g(\sigma, z).$$

Here we assume that the proliferation—modeled by the term $p(\sigma, z)\varphi(1 - \varphi/N)$—is affected by surgery-induced tissue damage and lactate concentration, which may increase apoptosis—represented by $-\varphi g(\sigma, z)$. Thus, the prescribed growth is not purely logistic, and a loss term is included (see, e.g., [Roc+08]). The terms $p(\sigma, z)$ and $g(\sigma, z)$ are, respectively, a proliferation rate and an apoptosis rate, and they may both depend on σ and z. Finally, χ_1 is the assigned concentration of a cytotoxic drug, which inhibits tumor proliferation. As for equation (1.3.14b), the lactate sources are similar to those introduced in [Aub+05] (see also [Gui+11; Lah+13; Che+22]). The term S collects the production of lactate in cells by glycolysis, the consumption of lactate by metabolism, and the diffusion of lactate in neighboring regions. It is affected by the presence of the lactate targeting drug χ_2. The term K accounts for lactate exchanges between tumor cells and the surroundings. We define it as

$$K(\varphi, \sigma, z) \coloneqq \frac{k_1(\varphi, z)\sigma}{k_2(\varphi, z) + \sigma},$$

where k_1 is the maximum transport rate of lactate through the monocarboxylate transporters, and k_2 is the modified Michaelis–Menten constant. We assume that k_1, k_2, and S are possibly dependent on the other variables φ and z.

1.4 Outline of the thesis

The thesis is organized as follows. In Chapter 2, we introduce the notation used throughout the manuscript and recall the main mathematical tools required for our proofs. The core of the thesis is contained in Chapters 3 to 5, where the three models presented above are investigated from a rigorous analytical perspective, with particular attention to the well-posedness of the corresponding initial and boundary value problems. Chapter 3 is devoted to the nonisothermal Caginalp-type model introduced in Section 1.3.1, Chapter 4 addresses the Cahn–Hilliard-type model with damage effects presented in Section 1.3.2, and Chapter 5 focuses on the brain tumor model discussed in Section 1.3.3. For the first phase field system (1.3.1)–(1.3.3), we establish existence, uniqueness, and regularity of the solutions. In the second initial-boundary value problem (1.3.5)–(1.3.7), we are able to prove the existence of solutions, but establishing uniqueness turns out to be challenging. Thus, after discussing the difficulties we face, we introduce a slightly modified version of the model, for which well-posedness can be rigorously demonstrated. Although this does not solve the original problem, it provides valuable insight and a potential direction for further analysis. Finally, for the Fisher–KPP brain tumor system (1.3.14)–(1.3.16), we also consider an associated optimal control problem, analysing the existence of optimal controls and deriving first-order necessary conditions.

The existence of weak solutions is usually proved by constructing suitable approximations of the PDE system, often via time discretization or a Faedo–Galerkin scheme, possibly combined with a regularization of the nonlinear terms. This is followed by the derivation of a priori estimates that ensure sufficient compactness, allowing one to pass to the limit in the approximation parameter and recover a solution to the original problem. Fixed-point arguments can also be employed as an alternative strategy. Under stronger assumptions on the initial data and the assigned functions, one can further prove the existence of strong solutions, and it might be possible to establish continuous dependence on the data.

Once the problem is shown to be well-posed and sufficient regularity is obtained, an associated optimal control problem can be investigated. In this context, the therapies in the PDE system are treated as control functions, which are required to belong to a suitable admissible set ensuring that, for each fixed control, the corresponding Cauchy problem admits a unique solution. A first step is to prove the existence of an optimal control, i.e., a control that minimizes a cost functional depending on both the state variables and the control itself. This leads to a nonlinear, nonconvex minimization problem subject to PDE constraints, typically addressed via the direct method of the Calculus of Variations. A major subsequent objective is to derive first-order necessary optimality conditions, usually expressed in the form of a variational inequality.

Chapter 2

Mathematical preliminaries

2.1 Notation

For any Banach space $(X, \|\cdot\|_X)$ we employ $(X', \|\cdot\|_{X'})$ for its topological dual, and $\langle\cdot,\cdot\rangle_X$ for the dual pairing between X' and X. If X is an Hilbert space, we use $(\cdot,\cdot)_X$ for its internal product.

Through the whole thesis, Ω will be an open bounded domain in $\mathbb{R}^d$ with $d = 2, 3$ with boundary $\Gamma \coloneqq \partial\Omega$ at least Lipschitz continuous, and outward unit normal $\boldsymbol{\nu}$. We set $T > 0$ to be a fixed final time, and we introduce

$$Q \coloneqq \Omega \times (0, T), \qquad \Sigma \coloneqq \Gamma \times (0, T).$$

We denote the Lebesgue and Sobolev spaces over Ω respectively as $L^p(\Omega)$ and $W^{k,p}(\Omega)$. In the special case of $p = 2$, we set $H^k(\Omega) \coloneqq W^{k,2}(\Omega)$. We employ the notation $L^p(\Gamma)$ for the Lebesgue spaces over Γ, and $\mathcal{H}^{d-1}$ for the $(d-1)$-dimensional Hausdorff measure. With a slight abuse of notation, we do not always distinguish between scalar, vector, and matrix-valued function spaces. For instance, while $L^p(\Omega)$ typically denotes scalar-valued functions, we also use it for vector-valued functions, which more precisely belong to $L^p(\Omega;\mathbb{R}^n)$. Sometimes, for $p \in [1, +\infty)$, we will identify $L^p(Q)$ with $L^p(0, T; L^p(\Omega))$.

For convenience, we introduce the notation

$$V \coloneqq H^1(\Omega), \qquad H \coloneqq L^2(\Omega).$$

In this context, it is common practice to identify H with its dual space H' and as a subspace of V' through

$$\langle w, v\rangle_V = (w, v)_H$$

for every $w \in H$, and $v \in V$. This identification leads to the well-known Hilbert triplet $V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V'$ where the embeddings are dense and compact. Moreover, it leads to the well-known interpolation inequality

$$\|v\|_H \leq \|v\|_V^{1/2} \|v\|_{V'}^{1/2} \tag{2.1.1}$$

which holds for every $v \in V$. We introduce the (generalized) mean value of $v' \in V'$ as

$$\langle v' \rangle \coloneqq \frac{1}{|\Omega|}\langle v', 1\rangle_V$$

for every $v' \in V'$. Obviously, it coincides with the usual mean value if $v \in H$ and, by extension, if $v \in L^1(\Omega)$. We employ

$$\dot{V}, \qquad \dot{H}, \qquad \dot{V}'$$

for the closed subspace respectively of V, H, V' of elements with zero mean value, and

$$V_0 \coloneqq H_0^1(\Omega),$$

where $H_0^1(\Omega)$ represents the set of $H^1(\Omega)$ functions with zero trace at the boundary. Additionally, we define

$$W \coloneqq \{v \in H^2(\Omega) \,|\, \partial_{\boldsymbol{\nu}} v = 0 \text{ on } \partial\Omega\}, \qquad W_0 \coloneqq H^2(\Omega) \cap V_0$$

which are, respectively, the subspace of $H^2(\Omega)$ of functions with zero normal derivative at the boundary, and with zero trace at the boundary. In both cases, the natural norm induced by $H^2(\Omega)$ is denoted by $\|\cdot\|_W$. Finally, we introduce the notation

$$Z \coloneqq W^{1,p}(\Omega).$$

The norm of the Bochner space $W^{k,p}(0,T;X)$ is indicated as $\|\cdot\|_{W^{k,p}(X)}$, omitting the time interval $(0,T)$ for the sake of brevity. If the final time differs from T, it will be written explicitly to avoid ambiguity. With the notation $C^0([0,T];X)$, we mean the space of continuous X-valued functions, while with $C^0_w([0,T];X)$ we mean the space of functions in $L^\infty(0,T;X)$ that are weakly continuous from $[0,T]$ to X.

As is customary, we use C to represent a generic constant depending only on the problem data and whose value might change from line to line. If we want to highlight a dependence on a certain parameter, we use it as a subscript (e.g., C_τ is a constant that depends on τ, C_0 is a constant that depends on the initial data, etc.).

Finally, a brief remark on notational conflicts. Despite the effort to maintain consistency, in a few minor cases, this was not possible. In some instances, the notation was adapted, while in others it was deliberately retained to align with the classical literature or with the published papers on which this thesis is based. In any case, such overlaps never occur within the same chapter, ensuring that the exposition remains unambiguous.

2.2 Continuous and compact Sobolev embeddings

The Sobolev embedding theorems assert that under suitable conditions, the Sobolev spaces $W^{k,p}(\Omega)$ are embedded continuously or compactly into other function spaces. These results depend on the dimension d, the order of differentiation k, and the integrability exponent p. In what follows, we summarize the main results that will be used throughout the text. For a more detailed exposition, see, e.g., [Leo17, Chapter 12].

Theorem 2.1 (Morrey–Sobolev–Gagliardo–Nirenberg)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $p \in [1, +\infty]$, and $k \in \mathbb{N}$.*

- *If $d > kp$, then*
$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in \left[p, \frac{pd}{d-kp}\right].$$
- *If $d = kp$, then*
$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, +\infty).$$
- *If $m \in \mathbb{N}$ and $\alpha \coloneqq k - \frac{d}{p} - m > 0$, then*
$$W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\Omega).$$

Theorem 2.2 (Rellich–Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $p \in [1, +\infty]$, and $k \in \mathbb{N}$.*

- *If $d > kp$, then*
$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in \left[p, \frac{pd}{d-kp}\right).$$
- *If $d = kp$, then*
$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [p, +\infty).$$
- *If $m \in \mathbb{N}$ and $0 < \alpha < k - \frac{d}{p} - m$, then*
$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{m,\alpha}(\Omega).$$

In particular, the most frequent continuous and compact embeddings that we are going to use are:

- if $d = 2$, then
$$H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [2, +\infty),$$
$$H^2(\Omega) \hookrightarrow C^{0,1}(\Omega);$$
- if $d = 3$, then
$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [2, 6], \qquad H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \forall q \in [2, 6),$$
$$H^2(\Omega) \hookrightarrow C^{0,1/2}(\Omega).$$

2.3 Basic inequalities

In our proofs, we frequently rely on a set of classical inequalities, which we list below for the reader's convenience.

- **The Young inequality.** Let a, b be two nonnegative numbers, and p, q be conjugate Hölder exponents, i.e.,

$$p \geq 1, \qquad q \geq 1, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

  Then, for every $\eta > 0$,

$$ab \leq \eta a^p + C_\eta b^q,$$

  where $C_\eta = q^{-1}(p\eta)^{-q/p}$. In particular, taking $p = q = 2$, it reads as

$$ab \leq \eta a^2 + \frac{1}{4\eta} b^2$$

  for every $\eta > 0$.

- **The Hölder inequality.** Let Ω be a measurable set, and $f_1, \dots, f_n$ be measurable functions on Ω. Then, for every $p_1, \dots, p_n, r \in [1, +\infty]$ such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r},$$

  it holds

$$\|f_1 \cdot \ldots \cdot f_n\|_{L^r} \leq \|f_1\|_{L^{p_1}} \cdot \ldots \cdot \|f_n\|_{L^{p_n}}.$$

- **The (generalized) Poincaré inequality.** Let Ω be a bounded Lipschitz domain, and $1 \leq p < +\infty$. Then, there exists a constant C_P (that depends only on the domain and on p), such that

$$\|v\|_{L^p} \leq C_P \left(\|\nabla v\|_{L^p} + \|v|_\Gamma\|_{L^p_\Gamma} \right)$$

  for every $v \in W^{1,p}(\Omega)$ with trace at the boundary $v|_\Gamma$.

- **The Poincaré–Wirtinger inequality.** Let Ω be a Lipschitz bounded domain, and $1 \leq p \leq +\infty$. Then, there exists a constant C_{PW} (that depends only on the domain and on p), such that

$$\|v - \langle v \rangle\|_{L^p} \leq C_{PW} \|\nabla v\|_{L^p}$$

  for every $v \in W^{1,p}(\Omega)$.

2.3.1 The Gronwall Lemma

A key tool in the study of evolution equations is the Gronwall Lemma, as it allows us to obtain a priori estimates and continuous dependence inequalities for the solutions of the PDE system. We recall its integral version, which will be frequently used throughout the thesis.

Lemma 2.3 (Integral Gronwall's inequality)**.** *Let f be a continuous function over $[0,T]$ satisfying the inequality*

$$f(t) \leq K + \int_0^t g(s) f(s)\, \mathrm{d}s \qquad \forall t \in [0,T],$$

for a nonnegative function $g \in L^1(0,T)$, and a nonnegative constant K. Then, it holds

$$f(t) \leq K e^{\int_0^t g(s)\, \mathrm{d}s} \qquad \forall t \in [0,T].$$

In the context of time-discrete schemes, a discrete analogue of Gronwall's inequality is often employed.

Lemma 2.4 (Discrete Gronwall's inequality)**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a real sequence satisfying*

$$x_n \leq K + \sum_{k=0}^{n-1} G_k x_k \qquad \forall n \in \mathbb{N}$$

for a constant K and a non-negative sequence $\{G_n\}_{n\in\mathbb{N}}$. Then, it holds

$$x_n \leq K \exp\left(\sum_{k=0}^{n-1} G_k\right)$$

for every $n \in \mathbb{N}$.

A proof can be found in [Cla87].

2.3.2 The Gagliardo–Nirenberg inequality

We recall the Gagliardo–Nirenberg interpolation inequality (see, e.g., [Nir59]).

Theorem 2.5 (Gagliardo–Nirenberg inequality)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain. Given*

$$r > q \geq 1, \qquad s > d\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \frac{1}{r} = \frac{\alpha}{q} + (1-\alpha)\left(\frac{1}{2} - \frac{s}{d}\right),$$

there exists a constant C such as for every $v \in H^s$, the following inequality holds true:

$$\|v\|_{L^r} \leq C \|v\|_{L^q}^{\alpha} \|v\|_{H^s}^{1-\alpha}.$$

In particular, we will employ it in the following special cases, also referred to as Ladyzhenskaya's inequalities.

Lemma 2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then, it exists a constant C such as, for every $v \in V$, it holds*

$$\|v\|_{L^4} \leq C\|v\|_H^{\frac{1}{2}}\|v\|_V^{\frac{1}{2}} \qquad \text{if} \quad d = 2, \tag{2.3.1}$$

$$\|v\|_{L^3} \leq C\|v\|_H^{\frac{1}{2}}\|v\|_V^{\frac{1}{2}}, \quad \|v\|_{L^4} \leq C\|v\|_H^{\frac{1}{4}}\|v\|_V^{\frac{3}{4}} \qquad \text{if} \quad d = 3. \tag{2.3.2}$$

From this result, employing the Young inequality, it is easy to obtain an inequality that we will extensively use throughout the thesis and, thus, is worth mentioning. For every $\eta > 0$ there exists a positive constant C_η such that, for $p = 3$ and $p = 4$, the following holds:

$$\|v\|_{L^p}^2 \leq \eta\|\nabla v\|_H^2 + C_\eta\|v\|_H^2, \tag{2.3.3}$$

for every $v \in V$, both in dimension $d = 2$ and $d = 3$.

2.3.3 The Ehrling Lemma

Another inequality we will employ is the following Ehrling's Lemma (see [LM12, Theorem 16.4, p. 102]).

Theorem 2.7 (Ehrling). *Let $(X_0, \|\cdot\|_{X_0})$, $(X, \|\cdot\|_X)$ and $(X_1, \|\cdot\|_{X_1})$ be Banach spaces with X_0 compactly embedded in X and X continuously embedded in X_1. Then, for every $\eta > 0$ there exists a $C_\eta > 0$ such that*

$$\|x\|_X \leq \eta\|x\|_{X_0} + C_\eta\|x\|_{X_1}$$

for every $x \in X_0$.

In our setting, we apply it for $V \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow V'$ and $p \in [2, 6)$, which is true in dimension $d = 2, 3$.

2.4 The Aubin–Lions Lemma

A crucial result in the study of nonlinear partial differential equations is the Aubin–Lions result, also known as the Aubin–Lions–Simon Lemma (see, e.g., [Sim86, Section 8, Corollary 4]). It gives a sufficient condition for compactness in a Bochner space $L^p(0, T; X)$, and is a powerful instrument to show that a subsequence of approximate solutions (in a sense that will be clarified in our proofs) strongly converges to a solution of the initial-boundary value problem under analysis.

Lemma 2.8 (Aubin–Lions). *Let X_0, X, and X_1 be be three Banach spaces such that*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1,$$

where X_0 is densely and compactly embedded in X, and X is densely and continuously embedded in X_1. Then, the following compact embeddings hold:

- *if* $p \in [1, +\infty)$ *and* $q \in [1, +\infty]$,

$$\left\{ v \in L^p(0,T;X_0), \frac{dv}{\mathrm{d}t} \in L^q(0,T;X_1) \right\} \hookrightarrow\hookrightarrow L^p(0,T;X),$$

- *if* $p = +\infty$ *and* $q \in (1, +\infty]$,

$$\left\{ v \in L^\infty(0,T;X_0), \frac{dv}{\mathrm{d}t} \in L^q(0,T;X_1) \right\} \hookrightarrow\hookrightarrow C^0([0,T];X).$$

Remark 2.9. Another key point in the study of weak solutions of certain partial differential equations is related to the study of their continuity. For example, this allows us to understand in which functional space the initial data must belong. A result in this direction is given by the second point of the Aubin–Lions Lemma, but it may be too strong to be applied. Let's consider two Banach spaces $X \hookrightarrow X_1$ where, as before, the embedding is dense and continuous. Moreover, we require that X is reflexive. A result from [Str66], shows that, if $v \in L^\infty(0,T;X) \cap C^0([0,T];X_1)$, it follows $v \in C^0_w([0,T];X)$. A trivial case in which it can be applied is when v belongs to

$$\left\{ v \in L^\infty(0,T;X), \frac{\mathrm{d}v}{\mathrm{d}t} \in L^1(0,T;X_1) \right\}.$$

2.5 Maximal monotone operators

In this section, we recall some fundamental definitions and results, following closely [Bré73]. Throughout, X denotes a Hilbert space. However, most of the concepts presented here can be extended to the more general setting where X is a Banach space (see, e.g., [Bar76], [Bar10]).

Definition 2.10. *A (multi-valued) operator A is an application from X into its power set 2^X, i.e., that assigns to every $x \in X$ a subset $Ax \subseteq X$. If, for every $x \in X$, Ax contains at most one element, A is said to be a single-valued operator. Its domain is defined as*

$$\mathcal{D}(A) := \{x \in X \mid Ax \neq \emptyset\},$$

and its image as

$$\mathcal{R}(A) := \bigcup_{x \in X} Ax.$$

The operator A can be identified with its graph, that is

$$\mathcal{G}(A) := \{(x,y) \in X \times X \mid x \in \mathcal{D}(A),\ y \in Ax\}.$$

Given two operators A, B, and two numbers $\eta, \delta \in \mathbb{R}$, it is natural to define their linear combination $\eta A + \delta B$ as

$$(\eta A + \delta B)x := \{\eta u + \delta v \mid u \in Ax,\ v \in Bx\}$$

for every $x \in X$, that has domain $\mathcal{D}(A) \cap \mathcal{D}(B)$. Moreover, we can define the inverse A^{-1} of A as follows:

$$y \in A^{-1}x \quad \iff \quad x \in Ay.$$

Clearly, its domain is $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$, and the graph of A^{-1} is the symmetric of the graph of A.

Definition 2.11. *An operator A is said to be monotone if*

$$(y_1 - y_2, x_1 - x_2)_X \geq 0$$

for every $x_1, x_2 \in X$ and for every $y_1 \in Ax_1, y_2 \in Ax_2$.

Over the set of operators defined on X, we can define a partial order based on the relation of graph inclusion, that is $A \subseteq B$ if and only if $\mathcal{G}(A) \subseteq \mathcal{G}(B)$. Among monotone operators, this relation is also inductive, meaning that given a nonempty chain, it surely has a maximal element. This justifies the following definition.

Definition 2.12. *A monotone operator A is said to be maximal monotone if it is not strictly included in any other monotone operator B, i.e., $A \subseteq B$ implies that $A = B$.*

Equivalently, an operator A is maximal monotone if it is monotone, and

$$(y - v, x - u)_X \geq 0 \quad \forall (u, v) \in \mathcal{G}(A)$$

implies that $(x, y) \in \mathcal{G}(A)$. Other useful characterizations are given by the result below (see [Bré73, Proposition 2.2, p. 23]).

Proposition 2.13. *The following statements are equivalent.*

i. *A is a maximal monotone operator.*

ii. *A is monotone and $\mathcal{R}(\mathrm{Id} + \mathrm{A}) = \mathrm{X}$.*

iii. *For all $\varepsilon > 0$, $J_\varepsilon := (\mathrm{Id} + \varepsilon \mathrm{A})^{-1}$ is defined on all X, and non-expansive, i.e.,*

$$\|J_\varepsilon x_1 - J_\varepsilon x_2\|_X \leq \|x_1 - x_2\|_X$$

for all $x_1, x_2 \in X$.

The operator J_ε is called the *resolvent operator*, since it associates with every $f \in X$ the unique solution $x \in \mathcal{D}(A)$ of the inclusion

$$x + \varepsilon Ax \ni f.$$

Notice that, being non-expansive, J_ε is obviously single-valued. Among the several crucial properties of maximal monotone operators, first of all, we recall that, for every $x \in \mathcal{D}(A)$, its image Ax is a closed and convex subset of X. This implies that the operator

$$A^0 x := P_{Ax}(0) = \arg\min_{y \in Ax} \|y\|_X \qquad \forall x \in \mathcal{D}(A)$$

is well defined. Here, P_{Ax} is the projection on the set Ax, so $A^0 x$ is the element of minimal norm of Ax. Another key result that we will use several times throughout the thesis is the following one (see [Bré73, Proposition 2.5, p. 27]).

Proposition 2.14. *Let A be a maximal monotone operator, and let $\{(x_n, y_n)\}_n \subseteq \mathcal{G}(A)$ be such that*

$$x_n \rightharpoonup x, \qquad y_n \rightharpoonup y, \qquad \limsup_{n\to+\infty} (x_n, y_n)_X \le (x, y)_X.$$

Then, $(x, y) \in \mathcal{G}(A)$ and $\lim_{n\to+\infty}(x_n, y_n)_X = (y, x)_X$.

In particular, given a sequence $\{(x_n, y_n)\}_n \subseteq \mathcal{G}(A)$ such that

$$x_n \to x, \qquad y_n \rightharpoonup y,$$

Proposition 2.14 can be applied, leading to the fact that the limit $(x, y) \in \mathcal{G}(A)$. Thus, the *strong-weak closure* of the graph of maximal monotone operators follows.

2.5.1 The Yosida approximation

Maximal monotone operators appear in nonlinear evolution equations, and significantly increase technical difficulties because, in general, they are multi-valued and may lack regularity. A powerful tool to overcome these issues is their Yosida approximation that, as we will see, is single-valued and Lipschitz continuous..

Definition 2.15. *For every $\varepsilon > 0$, the Yosida approximation of A of regularizing parameter ε is defined as*

$$A_\varepsilon := \frac{1}{\varepsilon}(\mathrm{Id} - \mathrm{J}_\varepsilon).$$

Notice that A_ε is a single-valued operator defined on all X. Moreover, it enjoys the following properties (see [Bré73, Proposition 2.6, p. 28]).

Proposition 2.16. *Let A be a maximal monotone operator on X and $\varepsilon > 0$. Then,*

i. A_ε is a maximal monotone operator.

ii. A_ε is Lipschitz continuous with Lipschitz constant $1/\varepsilon$.

iii. For every $x \in \mathcal{D}(A)$, then

$$\|A_\varepsilon x\|_X \le \|A^0 x\|_X.$$

Moreover, if $\varepsilon \to 0^+$, then

$$\|A_\varepsilon x\|_X \uparrow \|A^0 x\|_X \quad \text{and} \quad A_\varepsilon x \to A^0 x.$$

iv. For every $x \notin \mathcal{D}(A)$, if $\varepsilon \to 0^+$, then $\|A_\varepsilon x\|_X \uparrow +\infty$.

2.5.2 Subdifferentials of convex functions

An important class of maximal monotone operators is given by the subdifferentials of proper, convex, and lower semicontinuous functions. Subdifferentials arise naturally in convex analysis and play a central role in various applications, including optimization and evolution equations. In the following, we recall their definition and review the main properties relevant to our setting.

Definition 2.17. *Let $\phi : X \to (-\infty, +\infty]$ be a proper and convex function, i.e.,*

$$\mathcal{D}(\phi) \coloneqq \{x \in X \mid \phi(x) < +\infty\} \neq \emptyset,$$

and it satisfies

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all $x, y \in X$, and for all $t \in [0,1]$. The subdifferential of ϕ is the multi-valued operator $\partial\phi : X \to 2^X$ defined as follows

$$\partial\phi(x) \coloneqq \{\xi \in X \mid \phi(y) \geq \phi(x) + (\xi, y - x)_X \ \forall y \in X\}$$

for all $x \in X$.

The relevance of subdifferentials in minimization problems is quite clear, since

$$\phi(x) = \min_{y \in X} \phi(y) \quad \text{if and only if} \quad 0 \in \partial\phi(x).$$

The main result we are interested in is the following.

Lemma 2.18. *Let $\phi : X \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function, which means that $\liminf_{n\to+\infty} \phi(x_n) \geq \phi(x)$ if $x_n \to x$. Then, its subdifferential $\partial\phi$ is a maximal monotone operator.*

As we will frequently deal with operators of this form, understanding their Yosida approximation becomes particularly important.

Definition 2.19. *Let $\phi : X \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function. For every $\varepsilon > 0$, we define the Moreau–Yosida approximation of regularizing parameter ε as*

$$\phi_\varepsilon(x) \coloneqq \min_{y \in X} \left\{ \frac{1}{2\varepsilon} \|y - x\|_X^2 + \phi(y) \right\}.$$

The Moreau–Yosida approximation of ϕ enjoys several good properties, which link it with the Yosida approximation of $\partial\phi$ (see [Bré73, Proposition 2.11, p. 39]).

Lemma 2.20. *Let $\phi : X \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function. Then, the following statements hold.*

i. For every $x \in X$, $\phi_\varepsilon(x) = \dfrac{\varepsilon}{2}\|(\partial\phi)_\varepsilon x\|_X + \phi(J_\varepsilon x)$, where $(\partial\phi)_\varepsilon$ is the Yosida approximate of $\partial\phi$.

ii. ϕ_ε *is well defined, convex, and Fréchet differentiable in* X.

iii. The Fréchet derivative of ϕ_ε *coincides with* $(\partial\phi)_\varepsilon$.

iv. For every $x \in X$, $\phi_\varepsilon(x) \le \phi(x)$, *and* $\phi_\varepsilon(x) \uparrow \phi(x)$ *as* $\varepsilon \to 0^+$.

A meaningful example that we are going to employ throughout the thesis, is obtained considering a proper, convex, and lower semicontinuous function $\phi : \mathbb{R} \to (-\infty, +\infty]$, and its realization in H, defined naturally as $\Phi : H \to (-\infty, +\infty]$ such that

$$\Phi(v) := \begin{cases} \displaystyle\int_\Omega \phi(v)\,\mathrm{d}x & \text{if } \phi(v) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

for every $v \in H$. Then, Φ is proper, convex, and lower semicontinuous. Moreover, it holds that

$$\xi \in \partial\Phi(v) \quad \text{if and only if} \quad \xi(x) \in \partial\phi(v(x)) \text{ for a.e. } x \in \Omega.$$

Furthermore, the Moreau–Yosida approximations of parameter ε of the previous functions are linked by the following relation:

$$\Phi_\varepsilon(v) = \int_\Omega \phi_\varepsilon(v)\,\mathrm{d}x.$$

In light of all these properties, with a slight abuse of notation, we will write ϕ instead of Φ and $\partial\phi$ instead of $\partial\Phi$. For more details, the interested reader may refer to [Bré73, Proposition 2.16, p. 47].

2.5.3 The p-Laplace operator with homogeneous Neumann condition

Let $p \ge 2$ and define

$$\Phi_p : H \to [0, +\infty], \qquad \Phi_p(v) := \begin{cases} \displaystyle\frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x & \text{if } v \in Z, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, it is easy to show that Φ_p has domain $\mathcal{D}(\Phi_p) = Z$ and it is proper, convex, and lower semicontinuous on H. Hence, its subdifferential $-\Delta_p := \partial\Phi_p$ is a maximal monotone operator. Moreover, every v in the natural domain

$$\mathcal{D}(-\Delta_p) = \{v \in Z \,:\, -\Delta_p v \in H, \quad |\nabla v|^{p-2}\nabla v \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\},$$

it satisfies

$$\int_\Omega -\Delta_p v\,\omega\,\mathrm{d}x = \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\omega\,\mathrm{d}x$$

for every $\omega \in \mathcal{D}(\Phi_p)$. In particular,

$$-\Delta_p v = -\operatorname{div}(|\nabla v|^{p-2}\nabla v)$$

in the sense of distributions. Finally, the following regularity result holds (the interested reader can refer to [Sav98, Theorem 2, Remark 3.5]).

Lemma 2.21. *For all $0 < \delta < \frac{1}{p}$, the inclusion $\mathcal{D}(-\Delta_p) \subseteq W^{1+\delta,p}(\Omega)$ holds. Moreover, it exists $C_\delta > 0$ such that, for all $v \in W^{1+\delta,p}(\Omega)$,*

$$\|v\|_{W^{1+\delta,p}} \leq C_\delta(\|-\Delta_p v\|_H + \|v\|_H).$$

2.6 Mathematical linear viscoelasticity

As pointed out in the Introduction, Chapters 4 and 5 are set within the framework of linear viscoelasticity. The small displacement field $\boldsymbol{u}$ satisfies the (possibly quasi-static) balance of linear momentum

$$\kappa \partial_{tt} \boldsymbol{u} - \operatorname{div} \mathcal{T} = \boldsymbol{f},$$

where $\kappa \geq 0$ distinguishes between the inertial and quasi-static regimes. The stress tensor $\mathcal{T}$ is assumed to take the constitutive form

$$\mathcal{T} = \mathcal{C}\varepsilon(\boldsymbol{u}) + \mathcal{V}\varepsilon(\partial_t \boldsymbol{u}),$$

with $\mathcal{C}$ and $\mathcal{V}$ denoting the fourth-order elastic and viscous tensors, respectively, possibly depending on the point in the reference configuration Ω. Throughout Chapters 4 and 5 we will suppose that $\mathcal{C}$ is symmetric and positive definite, while $\mathcal{V}$ is symmetric and strictly positive definite, according to the following definition.

Definition 2.22. *Let $\mathcal{M} = (m_{hijk}) : \Omega \to \mathbb{R}^{d\times d\times d\times d}$ be a fourth-order tensor. We say that:*

i. *$\mathcal{M}$ is symmetric if*

$$m_{hijk}(x) = m_{ihjk}(x) = m_{jkhi}(x) \tag{2.6.1}$$

for a.e. $x \in \Omega$ and for each index $i, j, k, h = 1, \ldots, d$;

ii. *$\mathcal{M}$ is strictly positive definite (or uniformly elliptic) if there exists a positive constant C such that, for all $D \in \mathbb{R}^{d\times d}_{sym}$ and for a.e. $x \in \Omega$,*

$$\mathcal{M}(x)D : D \geq C|D|^2; \tag{2.6.2}$$

iii. *$\mathcal{M}$ is positive definite if*

$$\mathcal{M}(x)D : D \geq 0 \tag{2.6.3}$$

for all $D \in \mathbb{R}^{n\times n}_{sym}$ and for a.e. $x \in \Omega$.

Finally, let us recall a regularity result which we are going to frequently apply in the following.

Lemma 2.23. *Let Ω be a C^2 domain in $\mathbb{R}^d$ and $\mathcal{C} = (c_{hijk}) \in W^{1,\infty}(\Omega; \mathbb{R}^{d\times d\times d\times d})$ be a symmetric and strictly positive definite fourth-order tensor. Then, there exist $C_*, C^* > 0$ such that*

$$C_*\|\boldsymbol{u}\|_W \leq \|\operatorname{div}[\mathcal{C}\varepsilon(\boldsymbol{u})]\|_H \leq C^*\|\boldsymbol{u}\|_W$$

for every $\boldsymbol{u}$ in W_0.

For more details, cf. [MH94, Proposition 1.5, p. 318] and [Neč12, Lemma 3.2., p. 263].

Chapter 3

A nonisothermal phase field tumor growth model

The purpose of this chapter is to investigate the well-posedness and regularity of solutions to the initial–boundary value problem (1.3.1)–(1.3.3), following closely [CCR25]. We begin by introducing a suitable notion of weak solution and establishing its existence through a two-step approximation procedure, which combines a regularization of the potential with a Faedo–Galerkin discretization scheme. Under stronger assumptions on the initial data and the prescribed functions, we then prove the existence of strong solutions and establish their uniqueness by means of a continuous dependence result. It is worth noting that the heat source u in equation (1.3.1a), modeling hyperthermia therapy, can be regarded as a control function. Accordingly, this chapter can be interpreted as the analytical study of the state system of an optimal control problem, which is the subject of ongoing research in [CCR].

3.1 Hypotheses and main results

We consider a bounded domain Ω of class C^2 in $\mathbb{R}^d$ with $d = 2, 3$, and we fix a final time $T > 0$.

(H1) We assume that

$$\ell, \Lambda, \chi \text{ are positive constants,} \tag{3.1.1}$$

$$\tau \text{ and } \lambda_P, \lambda_A, \lambda_E, \lambda_C, \lambda_B, \lambda_D \text{ are real nonnegative constants.} \tag{3.1.2}$$

(H2) The assigned functions u and σ_B enjoy the regularity

$$u \in L^\infty(Q) \quad \text{with} \quad \|u\|_{L^\infty} \le M, \tag{3.1.3}$$

$$\sigma_B \in L^2(Q) \tag{3.1.4}$$

for a nonnegative constant M.

(H3) Regarding the nonlinearities $\mathbb{h}$ and $\mathbb{k}$, we require that

$$\mathbb{h}, \mathbb{k} \in C^{0,1}(\mathbb{R}), \text{ and} \tag{3.1.5}$$
$$0 \le \mathbb{h} \le \mathbb{h}', \ 0 \le \mathbb{k} \le \mathbb{k}', \tag{3.1.6}$$

for two nonnegative constants $\mathbb{h}'$, $\mathbb{k}'$.

(H4) We consider a potential $\Psi = \widehat{\beta} + \widehat{\pi}$ split into the sum of a convex part and a nonconvex perturbation. Explicitly, we suppose

$$\widehat{\beta}, \widehat{\pi} \in C^2(\mathbb{R}), \tag{3.1.7}$$
$$\widehat{\beta} \text{ is convex and nonnegative, with } \widehat{\beta}(0) = 0. \tag{3.1.8}$$

Moreover, introducing the notation $\beta := \widehat{\beta}'$ and $\pi := \widehat{\pi}'$, the following growth conditions hold:

$$|\beta(r)| \le C_\beta(\widehat{\beta}(r) + 1), \tag{3.1.9}$$
$$|\pi'(r)| \le C_\pi \tag{3.1.10}$$

for all $r \in \mathbb{R}$, where C_β, C_π are given nonnegative constants. We also point out that

$$\beta(0) = 0. \tag{3.1.11}$$

Remark 3.1. It follows that $\widehat{\pi}$ has at most quadratic growth and $\widehat{\beta}$ has at most exponential growth. A meaningful example of Ψ covered by our hypotheses is the regular quartic potential (1.2.3). Notice that the logarithmic potential (1.2.2), as well as the double obstacle potential, is instead excluded. In particular, we are not able to guarantee that the order parameter φ assumes values in the physically relevant interval $[-1, 1]$.

Remark 3.2. We observe that the assumption that $\widehat{\beta}$ attains its minimum value 0 at 0 (see (3.1.8) and (3.1.11)) is not restrictive. Indeed, if this were not the case, one could modify $\widehat{\beta}$ by subtracting its tangent line at 0, so that the new function $\widehat{\beta}$ remains convex and nonnegative, and additionally attains its minimum 0 at 0. Moreover, there is no issue in adding a linear term to $\widehat{\pi}$; doing so still preserves the Lipschitz continuity of π.

Definition 3.3. *We define a weak solution of the PDE system* (1.3.1)–(1.3.3) *as a quadruple* $(\theta, \varphi, \mu, \sigma)$ *with the regularity*

$$\theta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V), \tag{3.1.12}$$
$$\varphi \in H^1(0,T;V') \cap C^0_w([0,T];V) \cap L^2(0,T;W), \tag{3.1.13}$$
$$\mu \in L^2(0,T;V), \tag{3.1.14}$$
$$\sigma \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{3.1.15}$$

such that equations (1.3.1a)–(1.3.1d) *and the boundary conditions* (1.3.2) *are satisfied in the following variational sense*

$$\langle \partial_t(\theta + \ell\varphi), v\rangle_V + \int_\Omega \nabla\theta \cdot \nabla v \,\mathrm{d}x = \int_\Omega uv \,\mathrm{d}x, \tag{3.1.16a}$$

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu \cdot \nabla v \,\mathrm{d}x = \int_\Omega (\lambda_P\sigma - \lambda_A - \lambda_E\theta)\mathbb{h}(\varphi) v \,\mathrm{d}x, \tag{3.1.16b}$$

$$\begin{aligned}&\langle \tau\partial_t\varphi, v\rangle_V + \int_\Omega \nabla\varphi \cdot \nabla v \,\mathrm{d}x \\ &\quad + \int_\Omega (\beta(\varphi) + \pi(\varphi) - \chi\sigma - \Lambda\theta)\, v \,\mathrm{d}x = \int_\Omega \mu v \,\mathrm{d}x,\end{aligned} \tag{3.1.16c}$$

$$\begin{aligned}&\langle \partial_t\sigma, v\rangle_V + \int_\Omega \nabla\sigma \cdot \nabla v \,\mathrm{d}x - \chi\int_\Omega \nabla\varphi \cdot \nabla v \,\mathrm{d}x \\ &\quad = \int_\Omega (-\lambda_C\sigma\mathbb{h}(\varphi) + \lambda_B(\sigma_B - \sigma) - \lambda_D\sigma\mathbb{k}(\theta))\, v \,\mathrm{d}x,\end{aligned} \tag{3.1.16d}$$

a.e. in $(0, T)$ *for all* $v \in V$*, and the initial data* (1.3.3)

$$\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \tag{3.1.17}$$

a.e. in Ω*.*

Remark 3.4. An equivalent definition of weak solution is obtained by integrating in time the equalities in (3.1.16) over the interval $(0, T)$ and choosing test functions $v \in L^2(0, T; V)$. In addition, we point out that (3.1.16c) and (3.1.16d) can be equivalently rewritten as equations that hold almost everywhere and are complemented by homogeneous Neumann boundary conditions as in (1.3.2).

Theorem 3.5 (Existence of weak solutions)**.** *Assume that the set of hypotheses* (H1)–(H4) *holds and that the initial data satisfy*

$$\theta_0 \in H, \quad \varphi_0 \in V, \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega), \quad \sigma_0 \in V. \tag{3.1.18}$$

Then, the PDE system (1.3.1)–(1.3.3) *admits at least a weak solution* $(\theta, \varphi, \mu, \sigma)$ *with the additional regularity*

$$\tau^{1/2}\varphi \in H^1(0, T; H), \tag{3.1.19}$$

which satisfies the estimate

$$\begin{aligned}&\|\theta\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} + \|\varphi\|_{H^1(V')\cap L^\infty(V)\cap L^2(W)} + \|\tau^{1/2}\varphi\|_{H^1(H)} \\ &\quad + \|\beta(\varphi)\|_{L^2(H)} + \|\mu\|_{L^2(V)} + \|\sigma\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \leq C_1\end{aligned} \tag{3.1.20}$$

for a constant $C_1 > 0$ *that depends on* M *and on the other problem data.*

If we improve the regularity of the initial data, we prove that the system has a more regular solution, according to the following result.

Theorem 3.6 (Regularity)**.** *Assume that hypotheses (H1)–(H4) are satisfied, and that the initial data enjoy*

$$\theta_0 \in V \cap L^\infty(\Omega), \quad \varphi_0 \in W \cap H^3(\Omega), \quad \sigma_0 \in V. \tag{3.1.21}$$

Then, there exists a weak solution $(\theta, \varphi, \mu, \sigma)$ *with the additional regularity*

$$\theta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \tag{3.1.22}$$
$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.1.23}$$
$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;W). \tag{3.1.24}$$

In particular, $(\theta, \varphi, \mu, \sigma)$ *is a strong solution, i.e., it satisfies system* (1.3.1) *a.e. in* Q*. Moreover, there exists a constant* $C_2 > 0$ *that depends on* M *and on the other problem data, such that the following estimate holds*

$$\begin{aligned} &\|\theta\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)\cap L^\infty(Q)} + \|\varphi\|_{W^{1,\infty}(V')\cap H^1(V)\cap L^\infty(W)} \\ &\quad + \|\mu\|_{L^\infty(V)\cap L^2(W)} \le C_2. \end{aligned} \tag{3.1.25}$$

Remark 3.7. We point out that (3.1.21) in particular yields (3.1.18) and $\widehat{\beta}(\varphi_0) \in L^\infty(\Omega)$. Moreover, it is worth noting that, by standard Sobolev embedding results in dimensions $d = 2, 3$, the order parameter φ belongs to $L^\infty(Q)$.

The L^∞-bound for the component φ of the solution, combined with the local Lipschitz continuity of β, plays a crucial role in establishing a continuous dependence result, which, in particular, ensures the uniqueness property stated in the following theorem.

Theorem 3.8 (Continuous dependence)**.** *Suppose that hypotheses (H1)–(H4) are fulfilled, and that the initial data comply with* (3.1.21)*. Then, for every assigned functions* $\{u_i\}_{i=1,2}$ *satisfying* (3.1.3)*, and any pair of initial data* $\{(\theta_{0,i}, \phi_{0,i}, \sigma_{0,i})\}_{i=1,2}$ *satisfying* (3.1.21)*, if we denote by* $\{(\theta_i, \varphi_i, \mu_i, \sigma_i)\}_{i=1,2}$ *two corresponding strong solutions to* (1.3.1)*–*(1.3.3)*, the following continuous dependence inequality holds*

$$\begin{aligned} &\|\theta_1 - \theta_2\|_{L^\infty(H)\cap L^2(V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(H)\cap L^2(W)} \\ &\quad + \|\tau^{1/2}(\varphi_1 - \varphi_2)\|_{L^\infty(V)} + \|\sigma_1 - \sigma_2\|_{L^\infty(H)\cap L^2(V)} \\ &\le C_3\Big(\|\theta_{0,1} - \theta_{0,2}\|_H + \|\varphi_{0,1} - \varphi_{0,2}\|_H + \|\tau^{1/2}(\varphi_{0,1} - \varphi_{0,2})\|_V \\ &\quad + \|\sigma_{0,1} - \sigma_{0,2}\|_H + \|u_1 - u_2\|_{L^2(H)}\Big) \end{aligned} \tag{3.1.26}$$

for a positive constant C_3 *that depends on* M *and on the other problem data. Consequently, the strong solution found in Theorem 3.6 is unique.*

3.2 Existence of weak solutions

The existence of weak solutions is proved through two levels of approximation. First, we introduce an approximate problem in which β is replaced by a suitable Lipschitz continuous function β_ε. Second, to show the existence of weak solutions of the approximate PDE system, we employ a Faedo–Galerkin scheme. Finally, thanks to some a priori estimates, we separately pass to the limit, first in the Faedo–Galerkin scheme, and then as $\varepsilon \to 0$, finding a solution to the original problem. Notice that we need to introduce β_ε even if β is smooth due to the wide class of possible growth conditions that we are allowing: without it, we would not be able to pass to the limit in the corresponding term of the discretized system.

3.2.1 The approximate problem

For $\varepsilon \in (0,1)$, we define the Moreau–Yosida approximation of $\widehat{\beta}$ as

$$\widehat{\beta}_\varepsilon(r) := \min_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|s-r|^2 + \widehat{\beta}(s) \right\} \qquad \forall r \in \mathbb{R}, \tag{3.2.1}$$

and the Yosida regularization of β as

$$\beta_\varepsilon := (\widehat{\beta}_\varepsilon)'. \tag{3.2.2}$$

They enjoy the following properties:

(i) $\widehat{\beta}_\varepsilon$ is a $C^1(\mathbb{R})$ convex function with

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r) \qquad \text{for all } r \in \mathbb{R}, \tag{3.2.3}$$

(ii) β_ε is monotone increasing and Lipschitz continuous with $\beta_\varepsilon(0) = 0$ and with Lipschitz constant bounded by ε^{-1},

(iii) they satisfy inequality (3.1.9) with the same constant C_β, i.e.,

$$|\beta_\varepsilon(r)| \le C_\beta(\widehat{\beta}_\varepsilon(r) + 1) \quad \text{for all } r \in \mathbb{R}. \tag{3.2.4}$$

In view of (H4), properties (i)–(ii) are well known and can be found in Proposition 2.16 and Lemma 2.20. To prove (iii), we need to introduce the resolvent operator associated with β, defined as

$$J_\varepsilon(r) = (\mathrm{Id} + \varepsilon\beta)^{-1}(r), \tag{3.2.5}$$

i.e., $J_\varepsilon(r)$ satisfy $J_\varepsilon(r) + \varepsilon\beta(J_\varepsilon(r)) = r$ for all $r \in \mathbb{R}$. It can be proved (see Lemma 2.20) that $J_\varepsilon = \mathrm{Id} - \varepsilon\beta_\varepsilon$. It trivially follows that

$$\beta(J_\varepsilon(r)) = \frac{r - J_\varepsilon(r)}{\varepsilon} = \beta_\varepsilon(r) \quad \text{for all } r \in \mathbb{R}. \tag{3.2.6}$$

Moreover, again by Lemma 2.20, we know that

$$\widehat{\beta}_\varepsilon(r) = \frac{\varepsilon}{2}|\beta_\varepsilon(r)|^2 + \widehat{\beta}(J_\varepsilon(r)),$$

thus $\widehat{\beta}_\varepsilon(r) \geq \widehat{\beta}(J_\varepsilon(r))$. Putting these elements together and employing inequality (3.1.9), we obtain

$$|\beta_\varepsilon(r)| = |\beta(J_\varepsilon(r))| \leq C_\beta \left(1 + \widehat{\beta}(J_\varepsilon(r))\right) \leq C_\beta \left(1 + \widehat{\beta}_\varepsilon(r)\right),$$

so (iii) is proved.

The approximate problem is obtained from (1.3.1)–(1.3.3) replacing β with β_ε and its weak solutions are defined consequently as in Definition 3.3.

3.2.2 Existence of weak solutions for the approximate problem

As already anticipated, we prove the existence of weak solutions of the approximated system through a Faedo–Galerkin space discretization.

Faedo–Galerkin discretization. We introduce the nondecreasing sequence of eigenvalues $\{\gamma_j\}_{j\in\mathbb{N}_0}$ of the Laplace operator with homogeneous Neumann boundary conditions and the associated sequence of eigenvectors $\{e_j\}_{j\in\mathbb{N}_0}$, which is a complete orthonormal system in H, orthogonal in V and W. Explicitly, for every $j \in \mathbb{N}_0$,

$$\begin{cases} -\Delta e_j = \gamma_j e_j & \text{in } \Omega, \\ \partial_{\boldsymbol{\nu}} e_j = 0 & \text{on } \partial\Omega, \end{cases}$$

with the additional properties

$$(e_i, e_j)_= \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad (\nabla e_i, \nabla e_j)_= \gamma_i \delta_{ij},$$

for every $i, j \in \mathbb{N}_0$. Recall that, by standard elliptic regularity results, $\{e_j\}_{j\in\mathbb{N}_0}$ are smooth functions. We define

$$V^n := \operatorname{span}\{e_0, \dots, e_n\}$$

for every $n \in \mathbb{N}_0$. Then, $\{V^n\}_{n\in\mathbb{N}_0}$ is a nondecreasing sequence of subspaces, whose union is dense in V as well as in H. With our notation, $\gamma_0 = 0$, $e_0 = |\Omega|^{-1/2}$, and V^0 is the space of constant functions. We also introduce the projection of H onto V^n as

$$P^n(v) := \sum_{j=0}^{n} (v, e_j)_H e_j$$

for all $v \in H$. Notice that there exists a constant $C > 0$ such that

$$\|P^n(v)\|_X \leq C\|v\|_X \quad \text{for all } v \in X, \text{ where } X = H, V, W. \tag{3.2.7}$$

For every $n \in \mathbb{N}_0$, we aim at finding a quadruple $(\theta^n, \varphi^n, \mu^n, \sigma^n)$ of the form

$$\theta^n(x,t) = \sum_{j=1}^n \theta_j^n(t) e_j(x), \quad \varphi^n(x,t) = \sum_{j=1}^n \varphi_j^n(t) e_j(x),$$
$$\mu^n(x,t) = \sum_{j=1}^n \mu_j^n(t) e_j(x), \quad \sigma^n(x,t) = \sum_{j=1}^n \sigma_j^n(t) e_j(x)$$

such that

$$\langle \partial_t(\theta^n + \ell\varphi^n), v\rangle_V + \int_\Omega \nabla\theta^n \cdot \nabla v \,\mathrm{d}x = \int_\Omega uv \,\mathrm{d}x, \tag{3.2.8a}$$

$$\langle \partial_t\varphi^n, v\rangle_V + \int_\Omega \nabla\mu^n \cdot \nabla v \,\mathrm{d}x = \int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n) v \,\mathrm{d}x, \tag{3.2.8b}$$

$$\langle \tau\partial_t\varphi^n, v\rangle_V + \int_\Omega \nabla\varphi^n \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\beta_\varepsilon(\varphi^n) + \pi(\varphi^n) - \chi\sigma^n - \Lambda\theta^n)\, v \,\mathrm{d}x = \int_\Omega \mu^n v \,\mathrm{d}x, \tag{3.2.8c}$$

$$\langle \partial_t\sigma^n, v\rangle_V + \int_\Omega \nabla\sigma^n \cdot \nabla v \,\mathrm{d}x - \chi\int_\Omega \nabla\varphi^n \cdot \nabla v \,\mathrm{d}x = \int_\Omega (-\lambda_C\sigma^n\mathbb{h}(\varphi^n) + \lambda_B(\sigma_B - \sigma^n) - \lambda_D\sigma^n\mathbb{k}(\theta^n))\, v \,\mathrm{d}x, \tag{3.2.8d}$$

a.e. $t \in (0,T)$, for all $v \in V^n$, and that fulfill the initial conditions

$$\theta^n(0) = \theta_{0,n} := P^n(\theta_0), \quad \varphi^n(0) = \varphi_{0,n} := P^n(\varphi_0), \quad \sigma^n(0) = \sigma_{0,n} := P^n(\sigma_0). \tag{3.2.9}$$

Notice that, owing to the properties of the projection P^n, the following inequalities hold

$$\|\theta_{0,n}\|_H \le C\|\theta_0\|_H, \quad \|\varphi_{0,n}\|_V \le C\|\varphi_0\|_V, \quad \|\sigma_{0,n}\|_V \le C\|\sigma_0\|_V, \tag{3.2.10}$$

as well as the convergences

$$\theta_{0,n} \to \theta_0 \quad \text{strongly} \quad \text{in } H, \tag{3.2.11}$$
$$\varphi_{0,n} \to \varphi_0 \quad \text{strongly} \quad \text{in } V, \tag{3.2.12}$$
$$\sigma_{0,n} \to \sigma_0 \quad \text{strongly} \quad \text{in } V. \tag{3.2.13}$$

Here, we neglect the dependence on ε in the notation $(\theta^n, \varphi^n, \mu^n, \sigma^n)$ on purpose, for brevity.

Local-in-time existence. We test each equation of system (3.2.8)–(3.2.9) by e_j for $j = 0, \ldots, n$, obtaining the following ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta_j^n + \ell\varphi_j^n) + \gamma_j\theta_j^n = (u, e_j)_H, \tag{3.2.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_j^n + \gamma_j \mu_j^n = ((\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n)\mathbb{h}(\varphi^n), e_j)_H, \tag{3.2.14b}$$

$$\tau \frac{\mathrm{d}}{\mathrm{d}t}\varphi_j^n + \gamma_j \varphi_j^n + (\beta_\varepsilon(\varphi^n) + \pi(\varphi^n), e_j)_H - \chi \sigma_j^n - \Lambda \theta_j^n = \mu_j^n, \tag{3.2.14c}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sigma_j^n + (\gamma_j + \lambda_B)\sigma_j^n - \chi \gamma_j \varphi_j^n \\ &\quad = ((-\lambda_C \sigma^n \mathbb{h}(\varphi^n) + \lambda_B \sigma_B - \lambda_D \sigma^n \mathbb{k}(\theta^n)), e_j)_H, \end{aligned} \tag{3.2.14d}$$

$$\theta_j^n(0) = (\theta_{0,n}, e_j)_H, \quad \varphi^n(0) = (\varphi_{0,n}, e_j)_H, \quad \sigma_j^n(0) = (\sigma_{0,n}, e_j)_H. \tag{3.2.14e}$$

Notice that μ_j^n is an auxiliary variable and can be removed from the system by substituting μ_j^n, whose expression is given by equation (3.2.14c), into equation (3.2.14b). Then we can recover the expression for $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_j^n$ and replace it in equation (3.2.14a). Hence, we obtain a $3n$-equations first-order ODE system in the variables θ_j^n, φ_j^n, and σ_j^n for $j = 1, \ldots, n$ with locally Lipschitz continuous nonlinearities. Thus, the existence of $H^1(0, T^n)$ solutions follows from the Carathéodory existence theorem, for a certain $T^n \leq T$. The solution is not global because the nonlinearities are not globally Lipschitz continuous. Then, we retrieve $\mu_j^n \in L^2(0, T^n)$ by equation (3.2.14c). In the following, we will derive some a priori estimates independent of n that will allow us to extend the solution to the all interval $[0, T]$, and, at the same time, to recover enough compactness to pass to the limit as n goes to infinity. Keeping in mind that our final goal is passing to the limit as $\varepsilon \to 0$, we will also be careful in tracking the dependence on ε.

First a priori estimate. We integrate in time equation (3.2.8a) for a fixed $v \in V^n$, obtaining the equation

$$\int_\Omega (\theta^n + \ell \varphi^n) v \,\mathrm{d}x + \int_\Omega \nabla(1 *_t \theta^n) \cdot \nabla v \,\mathrm{d}x = \int_\Omega (1 *_t u) v + \int_\Omega (\theta_{0,n} + \ell \varphi_{0,n}) v \,\mathrm{d}x. \tag{3.2.15}$$

Here, the notation $*_t$ denotes the convolution with respect to time, i.e.,

$$1 *_t \theta^n := \int_0^t \theta^n(\cdot, s)\,\mathrm{d}s, \quad 1 *_t u := \int_0^t u(\cdot, s)\,\mathrm{d}s.$$

We take $v = \theta^n$ in equation (3.2.15) and multiply each side of the equality by a constant $R > 0$, fixed but yet to be determined. We choose $v = \varphi^n$ in (3.2.8b), $v = -\Delta\varphi^n$ in (3.2.8c), and $v = \sigma^n$ in (3.2.8d). We add the resulting equalities and, after some simplification, we obtain:

$$\begin{aligned} &R\int_\Omega |\theta^n|^2 \,\mathrm{d}x + \frac{R}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla(1 *_t \theta^n)|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi^n|^2 \,\mathrm{d}x + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\varphi^n|^2 \,\mathrm{d}x \\ &\quad + \int_\Omega |-\Delta\varphi^n|^2 \,\mathrm{d}x + \int_\Omega \beta_\varepsilon'(\varphi^n)|\nabla\varphi^n|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sigma^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla\sigma^n|^2 \,\mathrm{d}x \\ &= R\int_\Omega (1 *_t u)\theta^n \,\mathrm{d}x + R\int_\Omega (\theta_{0,n} + \ell\varphi_{0,n})\theta^n \,\mathrm{d}x - R\ell\int_\Omega \varphi^n \theta^n \,\mathrm{d}x \\ &\quad + \int_\Omega (\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n)\mathbb{h}(\varphi^n)\varphi^n \,\mathrm{d}x - \int_\Omega \pi(\varphi^n)(-\Delta\varphi^n)\,\mathrm{d}x + 2\int_\Omega \chi\sigma^n(-\Delta\varphi^n)\,\mathrm{d}x \end{aligned}$$

$$+\Lambda\int_\Omega \theta^n(-\Delta\varphi^n)\,\mathrm{d}x - \int_\Omega (\lambda_C \mathbb{h}(\varphi^n) + \lambda_D \mathbb{k}(\theta^n))\,|\sigma^n|^2\,\mathrm{d}x + \int_\Omega \lambda_B(\sigma_B - \sigma^n)\sigma^n\,\mathrm{d}x.$$

After applying the Hölder and the Young inequalities to the right-hand side, and rearranging some terms, we end up with

$$\begin{aligned}
&R\int_\Omega |\theta^n|^2\,\mathrm{d}x + \int_\Omega |-\Delta\varphi^n|^2\,\mathrm{d}x + \int_\Omega \beta_\varepsilon'(\varphi^n)|\nabla\varphi^n|^2\,\mathrm{d}x + \int_\Omega |\nabla\sigma^n|^2\,\mathrm{d}x + \int_\Omega \lambda_B|\sigma^n|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(R\int_\Omega |\nabla(1 *_t \theta^n)|^2\,\mathrm{d}x + \int_\Omega |\varphi^n|^2\,\mathrm{d}x + \tau\int_\Omega |\nabla\varphi^n|^2\,\mathrm{d}x + \int_\Omega |\sigma^n|^2\,\mathrm{d}x\right) \\
&= R\int_\Omega (1 *_t u + \theta_{0,n} + \ell\varphi_{0,n} - \ell\varphi^n)\,\theta^n\,\mathrm{d}x - \int_\Omega \lambda_E \mathbb{h}(\varphi^n)\varphi^n\theta^n\,\mathrm{d}x \\
&\quad + \int_\Omega (\lambda_P\sigma^n - \lambda_A)\mathbb{h}(\varphi^n)\varphi^n\,\mathrm{d}x + \int_\Omega (-\pi(\varphi^n) + 2\chi\sigma^n + \Lambda\theta^n)\,(-\Delta\varphi^n)\,\mathrm{d}x \\
&\quad - \int_\Omega (\lambda_C \mathbb{h}(\varphi^n) + \lambda_D \mathbb{k}(\theta^n))\,|\sigma^n|^2\,\mathrm{d}x + \int_\Omega \lambda_B\,\sigma_B\,\sigma^n\,\mathrm{d}x \\
&\le \left(\frac{R}{2} + \frac{1}{2} + \frac{\Lambda^2}{2}\right)\int_\Omega |\theta^n|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |-\Delta\varphi^n|^2\,\mathrm{d}x \\
&\quad + C_R\bigg(\int_\Omega |\theta_{0,n}|^2\,\mathrm{d}x + \int_\Omega |\varphi_{0,n}|^2\,\mathrm{d}x + \int_\Omega |1 *_t u|^2\,\mathrm{d}x + \int_\Omega |\varphi^n|^2\,\mathrm{d}x\bigg) \\
&\quad + C\bigg(\int_\Omega |\sigma_B|^2\,\mathrm{d}x + \int_\Omega |\sigma^n|^2\,\mathrm{d}x + 1\bigg),
\end{aligned}$$

where we have exploited the boundedness of $\mathbb{h}$ and $\mathbb{k}$ given by hypothesis (H2) and the Lipschitz continuity of π (cf. (3.1.10)). Now we choose $R = R(\Lambda)$ such that $R > 1 + \Lambda^2$, move the first two addends of the right-hand side to the left-hand side, and integrate in time over the interval $(0, t)$, for an arbitrary $t \le T^n \le T$. Moreover, we recall that β_ε' is nonnegative because β_ε is monotone. Thus, the corresponding integral is also nonnegative, and we can get rid of it. After renaming the constants, we have:

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega |\theta^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |-\Delta\varphi^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_\Omega |\nabla(1 *_t \theta^n)|^2\,\mathrm{d}x + \int_\Omega |\varphi^n|^2\,\mathrm{d}x + \tau\int_\Omega |\nabla\varphi^n|^2\,\mathrm{d}x + \int_\Omega |\sigma^n|^2\,\mathrm{d}x \\
&\le C_T\bigg(1 + \int_\Omega \left(|\theta_{0,n}|^2 + |\varphi_{0,n}|^2 + \tau|\nabla\varphi_{0,n}|^2 + |\sigma_{0,n}|^2\right)\,\mathrm{d}x \\
&\quad + \int_0^T\!\!\int_\Omega \left(|1 *_t u|^2 + |\sigma_B|^2\right)\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\!\!\int_\Omega \left(|\varphi^n|^2 + |\sigma^n|^2\right)\,\mathrm{d}x\,\mathrm{d}s\bigg).
\end{aligned}$$

Applying the Gronwall inequality leads to

$$\begin{aligned}
&\|\theta^n\|_{L^2(0,T^n;H)} + \|1 *_t \theta^n\|_{L^\infty(0,T^n;V)} + \|\varphi^n\|_{L^\infty(0,T^n;H)\cap L^2(0,T^n;W)} \\
&\quad + \|\tau^{1/2}\varphi^n\|_{L^\infty(0,T^n;V)} + \|\sigma^n\|_{L^\infty(0,T^n;H)\cap L^2(0,T^n;V)} \le C,
\end{aligned} \tag{3.2.16}$$

where C depends on the assigned data of the problem (in particular, on $\|1 *_t u\|_{L^2(H)}$), but not on T^n and ε.

Consequences of the first a priori estimate. From estimate (3.2.16), it trivially follows that

$$\|\langle\varphi^n\rangle\|_{L^\infty(0,T^n)} \leq C. \tag{3.2.17}$$

Moreover, testing equation (3.2.8b) with $v = |\Omega|^{-1}$ and estimating the right-hand side with its H-norm, we obtain

$$\|\langle\partial_t\varphi^n\rangle\|_{L^2(0,T^n)} \leq C\|(\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\|_{L^2(0,T^n;H)} \leq C. \tag{3.2.18}$$

Second a priori estimate. Since we are going to need it, we introduce the Neumann–Laplace operator restricted to the space $\dot{V}$ as

$$\begin{aligned}
&\mathcal{R} : \dot{V} \to \dot{V}' \text{ s.t.} \\
&\langle\mathcal{R}v, w\rangle_V := \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x \quad \forall w \in \dot{V}.
\end{aligned}$$

$\mathcal{R}$ is an isomorphism, and we denote its inverse by

$$\mathcal{N} := \mathcal{R}^{-1} : \dot{V}' \to \dot{V}.$$

The operators $\mathcal{R}$ and $\mathcal{N}$ enjoy several well-known properties, which we list here for the reader's convenience:

$$\langle v', \mathcal{N}w'\rangle_V = \langle w', \mathcal{N}v'\rangle_V = (\nabla\mathcal{N}w', \nabla\mathcal{N}v')_H \quad \text{for all } v', w' \in \dot{V}', \tag{3.2.19}$$

$$(w, v)_H = (w - \langle w\rangle, v)_H = \int_\Omega \nabla w \cdot \nabla\mathcal{N}v \,\mathrm{d}x \quad \text{for all } w \in V,\ v \in \dot{H}, \tag{3.2.20}$$

$$\|v\|_H \leq \|\nabla v\|_H^{1/2}\|\nabla\mathcal{N}v\|_H^{1/2} \quad \text{for every } v \in \dot{V}\ , \tag{3.2.21}$$

$$\begin{aligned}
&\langle\partial_t v'(t), \mathcal{N}v'(t)\rangle_V = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathcal{N}v'(t)\|_H^2 \\
&\qquad \text{for a.e. } t \in (0,T), \text{ for all } v' \in H^1(0,T;\dot{V}').
\end{aligned} \tag{3.2.22}$$

Since $\mathcal{N}$ is defined over the space $\dot{V}'$ but the functions we work with do not have, in general, zero mean value, it is useful to notice that the standard norms over W, V, H, and V' are equivalent to the following ones:

$$\begin{aligned}
\|v\|_W &\simeq (\|-\Delta v\|_H^2 + \langle v\rangle^2)^{1/2}, \\
\|v\|_V &\simeq (\|\nabla v\|_H^2 + \langle v\rangle^2)^{1/2}, \\
\|v\|_H &\simeq (\|v - \langle v\rangle\|_H^2 + \langle v\rangle^2)^{1/2}, \\
\|v\|_{V'} &\simeq (\|v - \langle v\rangle\|_{V'}^2 + \langle v\rangle^2)^{1/2} \simeq (\|\nabla\mathcal{N}(v - \langle v\rangle)\|_H^2 + \langle v\rangle^2)^{1/2}.
\end{aligned}$$

We preliminarily notice that if $v \in V^n$ and has zero mean value, then $\mathcal{N}v \in V^n$. Indeed, since v belongs to H, then $\mathcal{N}v$ belongs to W and the following equality obviously holds

$$-\Delta(\mathcal{N}v) = \sum_{j=0}^{+\infty}(-\Delta(\mathcal{N}v), e_j)_H e_j = \sum_{j=0}^{+\infty}\gamma_j(\mathcal{N}v, e_j)_H e_j.$$

On the other hand, we have that

$$-\Delta(\mathcal{N}v) = v = \sum_{j=0}^{n}(v, e_j)_H e_j.$$

Consequently, since these two expressions must coincide and $\gamma_j > 0$ for every $j \in \mathbb{N}$, we deduce that $(\mathcal{N}v, e_j)_H$ is equal to zero for every $j \geq n+1$. Thus, $\mathcal{N}v \in V^n$.
Now we have the instruments we need to derive the second a priori estimate. We take the difference of equation (3.2.8b) with its mean value and test it with $\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)$, then we add the resulting relation to equation (3.2.8c) tested with $(\varphi^n - \langle\varphi^n\rangle)$. In view of property (3.2.20), we note the cancellation of the terms involving μ^n as well as of the scalar products of a mean value and of $\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)$ or $(\varphi^n - \langle\varphi^n\rangle)$. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)|^2\,\mathrm{d}x + \tau\int_\Omega |\varphi^n - \langle\varphi^n\rangle|^2\,\mathrm{d}x\right)\\
&\quad + \int_\Omega |\nabla\varphi^n|^2\,\mathrm{d}x + \int_\Omega \beta_\varepsilon(\varphi^n)(\varphi^n - \langle\varphi^n\rangle)\,\mathrm{d}x\\
&= \int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)\,\mathrm{d}x\\
&\quad - \int_\Omega \pi(\varphi^n)(\varphi^n - \langle\varphi^n\rangle)\,\mathrm{d}x + \chi\int_\Omega \sigma^n(\varphi^n - \langle\varphi^n\rangle)\,\mathrm{d}x + \Lambda\int_\Omega \theta^n(\varphi^n - \langle\varphi^n\rangle).
\end{aligned}$$

Regarding the left-hand side, we observe that

$$\int_\Omega \beta_\varepsilon(\varphi^n)(\varphi^n - \langle\varphi^n\rangle)\,\mathrm{d}x \geq \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n)\,\mathrm{d}x - \int_\Omega \widehat{\beta}_\varepsilon(\langle\varphi^n\rangle)\,\mathrm{d}x,$$

due to the fact that β_ε is the subdifferential of $\widehat{\beta}_\varepsilon$. Then, we integrate in time over the interval $(0,t)$ for an arbitrary $t \leq T^n \leq T$, and estimate the terms on the right-hand side using the Hölder and the Young inequalities, obtaining

$$\begin{aligned}
&\frac{1}{2}\left(\int_\Omega |\nabla\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)|^2\,\mathrm{d}x + \tau\int_\Omega |\varphi^n - \langle\varphi^n\rangle|^2\,\mathrm{d}x\right)\\
&\quad + \int_0^t\int_\Omega |\nabla\varphi^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \widehat{\beta}_\varepsilon(\varphi^n)\,\mathrm{d}x\,\mathrm{d}s\\
&\leq \int_0^t\int_\Omega \widehat{\beta}_\varepsilon(\langle\varphi^n\rangle)\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_\Omega |\mathcal{N}(\varphi^n - \langle\varphi^n\rangle)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + C\left(\int_0^t\int_\Omega |\sigma^n|^2 + |\theta^n|^2 + |\varphi^n|^2 + |\varphi^n - \langle\varphi^n\rangle|^2\,\mathrm{d}x\,\mathrm{d}s + 1\right) =: I_1 + I_2 + I_3,
\end{aligned}$$

where we used the boundedness of $\mathbb{h}$ by hypothesis (H2), and the fact that π is Lipschitz continuous by hypothesis (H4). To handle I_1, we recall that $\widehat{\beta}_\varepsilon \leq \widehat{\beta}$, where $\widehat{\beta}$ is continuous by hypothesis (H4), and that, by (3.2.17), $\langle \varphi^n \rangle$ is uniformly bounded. Therefore, it follows that $I_1 \leq C$. We use the Poincaré inequality to treat I_2, taking into account that $\mathcal{N}(\varphi^n - \langle \varphi^n \rangle)$ has zero mean value. Finally, I_3 can be uniformly bounded thanks to (3.2.16) and (3.2.17). Thus, we have

$$\int_\Omega |\nabla \mathcal{N}(\varphi^n - \langle \varphi^n \rangle)|^2 \,\mathrm{d}x + \tau \int_\Omega |\varphi^n - \langle \varphi^n \rangle|^2 \,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\nabla \varphi^n|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t\!\!\int_\Omega \widehat{\beta}_\varepsilon(\varphi^n) \,\mathrm{d}x \,\mathrm{d}s$$
$$\leq C \left(\int_0^t\!\!\int_\Omega |\nabla \mathcal{N}(\varphi^n - \langle \varphi^n \rangle)|^2 \,\mathrm{d}x \,\mathrm{d}t + 1 \right)$$

from which, through the Gronwall Lemma, we obtain

$$\|\widehat{\beta}_\varepsilon(\varphi^n)\|_{L^1(\Omega\times(0,T^n))} \leq C. \tag{3.2.23}$$

Consequences of the second a priori estimate. From (3.2.23), recalling the $\widehat{\beta}_\varepsilon$ growth property (3.2.4), it trivially follows that

$$\|\beta_\varepsilon(\varphi^n)\|_{L^1(\Omega\times(0,T^n))} \leq C_\beta(1 + \|\widehat{\beta}_\varepsilon(\varphi^n)\|_{L^1(\Omega\times(0,T^n))}) \leq C. \tag{3.2.24}$$

Consequently, taking $v = |\Omega|^{-1}$ in equation (3.2.8c), recalling that $\langle \partial_t \varphi^n \rangle$ is estimated from (3.2.18) and thanks to the other estimates from (3.2.16), we deduce that

$$\|\langle \mu^n \rangle\|_{L^1(0,T^n)} \leq C. \tag{3.2.25}$$

Third a priori estimate. We take $v = (\Lambda/\ell)\theta^n$ in equation (3.2.8a), $v = \mu^n$ in (3.2.8b), $v = \partial_t \varphi^n$ in (3.2.8c), and $v = \partial_t \sigma^n$ in (3.2.8d). We sum the resulting equalities, noting a cancellation, and integrate in time over $(0,t)$ for an arbitrary $t \leq T^n \leq T$, obtaining:

$$\begin{aligned}
&\frac{\Lambda}{2\ell} \int_\Omega |\theta^n|^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla \varphi^n|^2 \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n) \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_\Omega |\nabla \sigma^n|^2 \,\mathrm{d}x + \frac{\Lambda}{\ell} \int_0^t\!\!\int_\Omega |\nabla \theta^n|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla \mu^n|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \tau \int_0^t\!\!\int_\Omega |\partial_t \varphi^n|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\partial_t \sigma^n|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&= \frac{\Lambda}{2\ell} \int_\Omega |\theta_{n,0}|^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla \varphi_{n,0}|^2 \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_{n,0}) \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_\Omega |\nabla \sigma_{n,0}|^2 \,\mathrm{d}x + \int_\Omega \widehat{\pi}(\varphi_{n,0}) - \widehat{\pi}(\varphi^n) \,\mathrm{d}x + \frac{\Lambda}{\ell} \int_0^t\!\!\int_\Omega u\theta^n \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \int_0^t\!\!\int_\Omega (\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n)\mathbb{h}(\varphi^n)\mu^n \,\mathrm{d}x \,\mathrm{d}s + \int_0^t\!\!\int_\Omega \chi \sigma^n \partial_t \varphi^n \,\mathrm{d}x \,\mathrm{d}s \\
&\quad - \int_0^t\!\!\int_\Omega \lambda_C \sigma^n \mathbb{h}(\varphi^n) \partial_t \sigma^n \,\mathrm{d}x \,\mathrm{d}s + \int_0^t\!\!\int_\Omega \lambda_B(\sigma_B - \sigma^n)\partial_t \sigma^n \,\mathrm{d}x \,\mathrm{d}s
\end{aligned}$$

$$- \int_0^t\int_\Omega \lambda_D \sigma^n \Bbbk(\theta^n)\partial_t \sigma^n \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \chi(-\Delta\varphi^n)\partial_t\sigma^n \,\mathrm{d}x\,\mathrm{d}s.$$

We aim to bound from above the right-hand side of this equality. First, we deal with the terms related to the initial data of the Galerkin discretized system. To do so, we recall that since they are projections of the original initial data, their H and V norms can be estimated with a constant independent of n. We notice that $\widehat{\pi}$ has at most quadratic growth thanks to hypothesis (H4). Moreover, the Moreau–Yosida approximation $\widehat{\beta}_\varepsilon$ satisfies

$$\widehat{\beta}_\varepsilon(r) \le \widehat{\beta}_\varepsilon(0) + \beta_\varepsilon(0)r + \frac{1}{2\varepsilon}r^2 \le \frac{1}{2\varepsilon}r^2 \quad \text{for every } r\in\mathbb{R}, \tag{3.2.26}$$

because of the definitions (3.2.1), (3.2.2) and the related property (ii) (cf. also (H4)), so here, just in order to bound the term

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{n,0})\,\mathrm{d}x,$$

we obtain an estimate which is not independent of ε. Most of the other terms on the right-hand side can be estimated simply through the Hölder and the Young inequalities, and the first a priori estimate (3.2.16), leading to

$$\begin{aligned}
&\frac{\Lambda}{2\ell}\int_\Omega |\theta^n|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi^n|^2\,\mathrm{d}x + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |\nabla\sigma^n|^2\,\mathrm{d}x + \frac{\Lambda}{\ell}\int_0^t\int_\Omega |\nabla\theta^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega |\nabla\mu^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tau\int_0^t\int_\Omega |\partial_t\varphi^n|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega |\partial_t\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le C_{0,\varepsilon} + C\int_\Omega \big(|\varphi^n|^2+1\big)\,\mathrm{d}x + \frac{\Lambda}{2\ell}\int_0^t\int_\Omega |u|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\Lambda}{2\ell}\int_0^t\int_\Omega |\theta^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\mu^n\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \chi\sigma^n\partial_t\varphi^n\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{1}{4}\int_0^t\int_\Omega |\partial_t\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_\Omega |\sigma^n|^2 + |\sigma_B|^2 + |-\Delta\varphi^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\le C_{0,\varepsilon} + C + \frac{1}{4}\int_0^t\int_\Omega |\partial_t\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\mu^n\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \chi\sigma^n\partial_t\varphi^n\,\mathrm{d}x\,\mathrm{d}s \\
&=: C_{0,\varepsilon} + C + \frac{1}{4}\int_0^t\int_\Omega |\partial_t\sigma^n|^2\,\mathrm{d}x\,\mathrm{d}s + I_4 + I_5.
\end{aligned} \tag{3.2.27}$$

At this point, we focus on the terms I_4 and I_5. To treat I_4, we employ the Hölder inequality and the fact that $\|\sigma^n\|_{L^\infty(0,T^n;H)}$ is uniformly bounded by (3.2.16). Then, we use the Poincaré, the Hölder, the Young inequalities and the previous estimates (3.2.16), (3.2.25). We have:

$$\begin{aligned}
I_4 &\le C\int_0^t (\|\sigma^n\|_H + \|\theta^n\|_H + 1)\|\mu^n\|_H \,\mathrm{d}s \le C\int_0^t (\|\theta^n\|_H + 1)\|\mu^n\|_H \,\mathrm{d}s \\
&\le C\int_0^t (\|\theta^n\|_H + 1)\|\nabla\mu^n\|_H \,\mathrm{d}s + C\int_0^t (\|\theta^n\|_H + 1)|\langle\mu^n\rangle| \,\mathrm{d}s \\
&\le \frac{1}{2}\int_0^t \|\nabla\mu^n\|_H^2 \,\mathrm{d}s + C + C\int_0^t |\langle\mu^n\rangle|\, \|\theta^n\|_H^2 \,\mathrm{d}s.
\end{aligned} \tag{3.2.28}$$

We handle the term I_5 integrating by parts with respect to time and then through the Hölder and Young inequalities. Explicitly, we have

$$\begin{aligned}
I_5 &= -\int_0^t\!\!\int_\Omega \chi\partial_t\sigma^n\varphi^n \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \chi\sigma^n\varphi^n \,\mathrm{d}x - \int_\Omega \chi\sigma_{n,0}\varphi_{n,0} \,\mathrm{d}x \\
&\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\partial_t\sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s + C\left(\int_0^t\!\!\int_\Omega |\varphi^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_\Omega |\sigma^n|^2 + |\varphi^n|^2 \,\mathrm{d}x + 1\right) \\
&\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\partial_t\sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s + C,
\end{aligned} \tag{3.2.29}$$

where in the last inequality we used the first a priori estimate (3.2.16). We return to (3.2.27), and make use of (3.2.28) and (3.2.29). Rearranging some terms and renaming the constants, we end up with

$$\begin{aligned}
&\int_\Omega |\theta^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla\varphi^n|^2 \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n) \,\mathrm{d}x + \int_\Omega |\nabla\sigma^n|^2 \,\mathrm{d}x \\
&\quad + \int_0^t\!\!\int_\Omega |\nabla\theta^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla\mu^n|^2 \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \tau\int_0^t\!\!\int_\Omega |\partial_t\varphi^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\partial_t\sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C_{0,\varepsilon} + C + C\int_0^t |\langle\mu^n\rangle|\, \|\theta^n\|_H^2 \,\mathrm{d}s.
\end{aligned}$$

Thanks to the Gronwall inequality and the fact that $\|\langle\mu^n\rangle\|_{L^1(0,T^n)}$ is uniformly bounded from (3.2.25), we deduce

$$\begin{aligned}
&\|\theta^n\|_{L^\infty(0,T^n;H)\cap L^2(0,T^n;V)} + \|\varphi^n\|_{L^\infty(0,T^n;V)} \\
&\quad + \|\tau^{1/2}\varphi^n\|_{H^1(0,T^n;H)} + \|\widehat{\beta}_\varepsilon(\varphi^n)\|_{L^\infty(0,T^n;L^1(\Omega))} \\
&\quad + \|\nabla\mu^n\|_{L^2(0,T^n;H)} + \|\sigma^n\|_{H^1(0,T^n;H)\cap L^\infty(0,T^n;V)} \le C_\varepsilon.
\end{aligned} \tag{3.2.30}$$

Consequences of the third a priori estimate. Owing to estimate (3.2.30) and exploiting the growth property (3.2.4), it is straightforward to infer that

$$\|\beta_\varepsilon(\varphi^n)\|_{L^\infty(0,T^n;L^1(\Omega))} \le C_\varepsilon. \tag{3.2.31}$$

Thus, proceeding as before and choosing $v = |\Omega|^{-1}$ in equation (3.2.8c), by comparing the terms we have that

$$\|\langle \mu^n \rangle\|_{L^2(0,T^n)} \leq C_\varepsilon, \tag{3.2.32}$$

thanks to (3.2.30). Then, by the Poincaré inequality, it follows that

$$\|\mu^n\|_{L^2(0,T^n;V)} \leq C_\varepsilon. \tag{3.2.33}$$

Next, we want to show that

$$\|\partial_t \varphi^n\|_{L^2(0,T^n;V')} \leq C_\varepsilon. \tag{3.2.34}$$

To do so, we consider an element $v \in V$ and proceed as follows

$$\begin{aligned}
|\langle \partial_t \varphi^n, v\rangle_V| &= |\langle \partial_t \varphi^n, P^n(v)\rangle_V| \\
&= \left| -\int_\Omega \nabla \mu^n \cdot \nabla \left[P^n(v)\right] \,\mathrm{d}x + \int_\Omega (\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n) \mathbb{h}(\varphi^n) P^n(v) \,\mathrm{d}x \right| \\
&\leq C \left(\|\nabla \mu^n\|_H + \|\sigma^n\|_H + \|\theta^n\|_H + 1\right) \|P^n(v)\|_V \\
&\leq C \left(\|\nabla \mu^n\|_H + \|\sigma^n\|_H + \|\theta^n\|_H + 1\right) \|v\|_V,
\end{aligned} \tag{3.2.35}$$

where, besides the usual calculations, we exploit the fact that $\partial_t \varphi^n$ satisfies equation (3.2.8b) and that, even though v does not belong to the space V^n of test functions, its projection does. From this inequality, we find that

$$\begin{aligned}
\|\partial_t \varphi^n\|^2_{L^2(0,T^n;V')} &= \int_0^{T^n} \|\partial_t \varphi^n\|^2_{V'} \,\mathrm{d}t \\
&\leq C \int_0^{T^n} \left(\|\nabla \mu^n\|^2_H + \|\sigma^n\|^2_H + \|\theta^n\|^2_H + 1\right) \,\mathrm{d}t \leq C_\varepsilon,
\end{aligned}$$

so (3.2.34) follows. Proceeding similarly, it is easy to prove that

$$\|\partial_t \theta^n\|_{L^2(0,T^n;V')} \leq C_\varepsilon. \tag{3.2.36}$$

Finally, we take $v = -\Delta \sigma^n$ in equation (3.2.8d) and integrate in time, obtaining

$$\begin{aligned}
&\|-\Delta \sigma^n\|^2_{L^2(0,T^n;H)} \,\mathrm{d}t \\
&\quad = -\int_0^{T^n} \int_\Omega \partial_t \sigma^n (-\Delta \sigma^n) \,\mathrm{d}x \,\mathrm{d}t + \chi \int_0^{T^n} \int_\Omega (-\Delta \varphi^n)(-\Delta \sigma^n) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \int_0^{T^n} \int_\Omega \left(-\lambda_C \sigma^n \mathbb{h}(\varphi^n) + \lambda_B(\sigma_B - \sigma^n) - \lambda_D \sigma^n \mathbb{k}(\theta^n)\right)(-\Delta \sigma^n) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad \leq C \left(\|\sigma^n\|_{H^1(0,T^n;H)} + \|-\Delta \varphi^n\|_{L^2(0,T^n;H)} + \|\sigma_B\|_{L^2(0,T^n;H)}\right) \|-\Delta \sigma^n\|_{L^2(0,T^n;H)} \\
&\quad \leq C_\varepsilon \|-\Delta \sigma^n\|_{L^2(0,T^n;H)},
\end{aligned}$$

where we used the Hölder inequality and the previously proved estimates (3.2.16) and (3.2.30). Thus, $\|-\Delta \sigma^n\|_{L^2(0,T^n;H)}$ is uniformly bounded and, by elliptic regularity, we have that

$$\|\sigma^n\|_{L^2(0,T^n;W)} \leq C_\varepsilon. \tag{3.2.37}$$

Global-in-time existence. Let $\overline{T}^n \le T$ denote the maximal existence time of the local-in-time solution $(\theta^n, \varphi^n, \mu^n, \sigma^n)$. Suppose, by contradiction, that $\overline{T}^n < T$. Thanks to the uniform (in n) a priori estimates established earlier and standard embedding results, by continuity the solution is defined at time $\overline{T}^n$ and satisfies the requirement on the initial values (cf. (3.2.10) and (3.1.18))

$$\|\theta^n(\overline{T}^n)\|_H + \|\varphi^n(\overline{T}^n)\|_V + \|\sigma^n(\overline{T}^n)\|_V \le C.$$

Therefore, we can take the solution at $t = \overline{T}^n$ as a new initial datum for the system of differential equations and extend the solution beyond $\overline{T}^n$ by continuity. This leads to a contradiction, since it implies that $\overline{T^n}$ is not maximal. We conclude that $\overline{T}^n = T$.

Passage to the limit as $n \to \infty$. In the previous steps, we proved that the solution of the Faedo–Galerkin discretized system satisfies the following bound

$$\|\theta^n\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} + \|\varphi^n\|_{H^1(V')\cap L^\infty(V)\cap L^2(W)} + \|\mu^n\|_{L^2(V)} + \|\sigma^n\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \le C_\varepsilon.$$

Thus, from standard compactness results (i.e., Banach–Alaoglu and Aubin–Lions theorems), we deduce that there exists a quadruple $(\theta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ satisfying the regularity of Definition 3.3 such that, for $n \to +\infty$, along a nonrelabelled subsequence, the following convergences hold:

$$\theta^n \to \theta_\varepsilon \quad \text{weakly-}* \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.2.38}$$
$$\text{strongly} \quad \text{in } L^2(0,T;H), \tag{3.2.39}$$
$$\text{a.e.} \quad \text{in } Q, \tag{3.2.40}$$
$$\varphi^n \to \varphi_\varepsilon \quad \text{weakly-}* \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.2.41}$$
$$\text{strongly} \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.2.42}$$
$$\text{a.e.} \quad \text{in } Q, \tag{3.2.43}$$
$$\mu^n \to \mu_\varepsilon \quad \text{weakly} \quad \text{in } L^2(0,T;V), \tag{3.2.44}$$
$$\sigma^n \to \sigma_\varepsilon \quad \text{weakly-}* \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{3.2.45}$$
$$\text{strongly} \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{3.2.46}$$
$$\text{a.e.} \quad \text{in } Q. \tag{3.2.47}$$

We integrate in time the equations (3.2.8a)–(3.2.8d) over the interval $(0,T)$, after choosing as a test function $P^n(v) \in L^2(0,T;V^n)$ for a generic fixed $v \in L^2(0,T;V)$. We recall that

$$P^n(v) \to v \quad \text{strongly} \quad \text{in } L^2(0,T;V). \tag{3.2.48}$$

All the linear terms pass to the limit thanks to the weak convergences (3.2.38), (3.2.41), (3.2.44), (3.2.45), and to the strong convergence (3.2.48). Let's discuss only the nonlinear ones. In the mass source term in equation (3.2.8b), we have that

$$(\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n) \quad \to \quad (\lambda_P\sigma_\varepsilon - \lambda_A - \lambda_E\theta_\varepsilon)\mathbb{h}(\varphi_\varepsilon)$$

weakly in $L^2(0,T;H)$. In fact, the term between brackets strongly converges in $L^2(0,T;H)$ and a.e. in Q; besides, $\mathbb{h}$ is continuous and bounded by hypothesis (H3) and $\varphi^n \to \varphi$ a.e. in Q. Thus, we have weak convergence in $L^2(0,T;H)$ by uniform boundedness, and a.e. convergence, with identification of the limit (see, e.g., [Lio69, Lemme 1.3, p. 12]). Regarding equation (3.2.8c), we notice that $\beta_\varepsilon + \pi$ is Lipschitz continuous. Thus, the strong convergence (3.2.42) is enough to pass to the limit. Finally, the term

$$-\lambda_C \sigma^n \mathbb{h}(\varphi^n) - \lambda_D \sigma^n \mathbb{k}(\theta^n) \quad \to \quad -\lambda_C \sigma_\varepsilon \mathbb{h}(\varphi_\varepsilon) - \lambda_D \sigma_\varepsilon \mathbb{k}(\theta_\varepsilon)$$

weakly in $L^2(0,T;H)$, because, as we similarly did before, $\mathbb{h}$ and $\mathbb{k}$ are continuous and bounded by hypothesis (H3) and σ^n, θ^n converges a.e. in Q by (3.2.43), (3.2.40). To conclude the proof, we only need to justify the fact that θ_ε, φ_ε, and σ_ε satisfy the initial conditions. On one hand, we observe that, along a subsequence, $\theta^n \to \theta_\varepsilon$, $\varphi^n \to \varphi_\varepsilon$, and $\sigma^n \to \sigma_\varepsilon$ strongly in $C^0([0,T];V')$ at least. Thus, $\theta^n(0) \to \theta_\varepsilon(0)$, $\varphi^n(0) \to \varphi_\varepsilon(0)$, and $\sigma^n(0) \to \sigma_\varepsilon(0)$ strongly in V'. On the other hand, $\theta^n(0) = P^n(\theta_0) \to \theta_0$, $\varphi^n(0) = P^n(\varphi_0) \to \varphi_0$, and $\sigma^n(0) = P^n(\sigma_0) \to \sigma_0$ strongly in H by (3.2.11)–(3.2.13). By uniqueness of the limit, we have

$$\theta_\varepsilon(0) = \theta_0, \quad \varphi_\varepsilon(0) = \varphi_0, \quad \sigma_\varepsilon(0) = \sigma_0.$$

3.2.3 A priori estimate uniform in ε

Notice that the first and the second a priori estimates and their consequences on the Galerkin level are already independent of ε, so they pass to the limit as $n \to +\infty$ by lower semicontinuity, and are satisfied by $(\theta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$. The ε-dependence starts to appear in the third a priori estimate, and then, starting from it, it spreads out. Thus, we need to re-perform it, focusing on the problematic term, which now becomes

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(0))\,\mathrm{d}x.$$

Taking into account that β_ε can be bounded by $\widehat{\beta}$ (see inequality (3.2.3)), and that $\widehat{\beta}(\varphi_0)$ is integrable by assumption (3.1.18), this integral is estimated as follows

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(0))\,\mathrm{d}x = \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0)\,\mathrm{d}x \le \int_\Omega \widehat{\beta}(\varphi_0)\,\mathrm{d}x < +\infty,$$

independently from ε. Thus, also the third a priori estimate and its consequences hold with a constant independent of ε. By summarizing, we have that

$$\begin{aligned} &\|\theta_\varepsilon\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} + \|\varphi_\varepsilon\|_{H^1(V')\cap L^\infty(V)\cap L^2(W)} \\ &\quad + \|\tau^{1/2}\varphi_\varepsilon\|_{H^1(H)} + \|\mu_\varepsilon\|_{L^2(V)} + \|\sigma_\varepsilon\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \le C \end{aligned} \tag{3.2.49}$$

for some constant C depending on M and on the problem data, but independent of ϵ. Now we can derive an additional estimate, proceeding by comparison in the version of (3.1.16c) written for β_ε (in place of β) and $(\theta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$. In fact, in view of the regularity of the solution, we can write the equivalent equation

$$\tau\partial_t\varphi_\varepsilon - \Delta\varphi_\varepsilon + \beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon) - \chi\sigma_\varepsilon - \Lambda\theta_\varepsilon = \mu_\varepsilon \qquad \text{a.e. in } Q \tag{3.2.50}$$

and then compare the terms in the light of (3.2.49). Hence, we can recover a uniform estimate for $\beta_\varepsilon(\varphi_\varepsilon)$, namely

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(H)} \leq C. \tag{3.2.51}$$

3.2.4 Passage to the limit as $\varepsilon \to 0$

From estimates (3.2.49), (3.2.51) and standard compactness results, there exists a quadruple $(\theta, \varphi, \mu, \sigma)$ satisfying the regularity properties in Definition 3.3 such that, as $\varepsilon \to 0$, along a nonrelabelled subsequence, the same convergences we had in (3.2.38)–(3.2.47) hold, with the additional

$$\beta_\varepsilon(\varphi_\varepsilon) \to \beta(\varphi) \qquad \text{weakly} \qquad \text{in } L^2(0,T;H). \tag{3.2.52}$$

In fact, since $\beta_\varepsilon(\varphi_\varepsilon)$ is uniformly bounded in $L^2(0,T;H)$ it converges, along a subsequence, to a certain $\xi \in L^2(0,T;H)$. Moreover, we know that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(0,T;H)$. Thus, we may identify ξ with $\beta(\varphi)$ because of the strong-weak closeness of the graph of the maximal monotone operator β (see Proposition 2.14). This is enough to pass to the limit in the approximate system, showing that the limit $(\theta, \varphi, \mu, \sigma)$ is a weak solution to the PDE system (1.3.1)–(1.3.3). Moreover, $(\theta, \varphi, \mu, \sigma)$ satisfies estimate (3.1.20).

3.3 Regularity

This section contains the proof of Theorem 3.6. To establish the existence of a more regular solution to our problem, we start again from the Faedo–Galerkin discretized system, taking advantage of the additional assumptions stated in (3.1.21). Then, we perform additional estimates and consequently obtain a limiting solution with the desired regularity.

Additional regularity of the discrete solution. We aim to check that

$$\varphi^n \in H^2(0,T;V^n), \quad \mu^n \in H^1(0,T;V^n). \tag{3.3.1}$$

We consider a fixed index $j = 0, \ldots, n$. As already noticed, μ_j^n is a silent variable in the discrete Cahn–Hilliard equation (3.2.14b)–(3.2.14c), and can be removed by substituting μ_j^n, whose expression in given by equation (3.2.14c), into equation (3.2.14b). This way, we obtain:

$$\begin{aligned}(1+\tau\gamma_j)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_j^n = ((\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n), e_j)_{-}\gamma_j^2\varphi_j^n \\ - \gamma_j(\beta_\varepsilon(\varphi^n) + \pi(\varphi^n), e_j)_{+}\chi\gamma_j\sigma_j^n + \Lambda\gamma_j\theta_j^n.\end{aligned} \tag{3.3.2}$$

Looking at the right-hand side, we already know that $-\gamma_j^2\varphi_j^n + \chi\gamma_j\sigma_j^n + \Lambda\gamma_j\theta_j^n$ belongs to the space $H^1(0,T)$. To conclude, since $e_j \in W \hookrightarrow L^\infty(\Omega)$, we only need to check that

$$(\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n) - \gamma_j(\beta_\varepsilon(\varphi^n) + \pi(\varphi^n)) \in H^1(0,T;L^1(\Omega)),$$

which is straightforward since $\mathbb{h}$, β_ε, and π are Lipschitz continuous, and σ^n, θ^n, and φ^n belong at least to $H^1(0,T;H)$. Thus, φ_j^n has the desired time-regularity and, consequently, φ^n satisfies (3.3.1). By comparison in equation (3.2.14c), it follows that $\mu_j^n \in H^1(0,T)$, so (3.3.1) is verified.

Fourth a priori estimate. First of all, we make an observation. In what follows, we will make use of the *a priori* estimates derived in the proof of Theorem 3.5. Recall that, at the Galerkin level, the first two estimates are uniform with respect to both n and ε, whereas the third one is uniform only in n. This is due to the fact that we were unable to bound the term

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{n,0})\,\mathrm{d}x$$

uniformly in ε. However, due to the stronger hypothesis on φ_0, now we can do it. Since φ_0 belongs to the space $W \cap H^3(\Omega)$, it holds that (cf. (3.2.7))

$$\|\varphi_{n,0}\|_W = \|P^n(\varphi_0)\|_W \le C\|\varphi_0\|_W, \tag{3.3.3}$$

$$\|\Delta\varphi_{n,0}\|_V = \|P^n(\Delta\varphi_0)\|_V \le C\|\Delta\varphi_0\|_V \le C\|\varphi_0\|_{W\cap H^3(\Omega)}, \tag{3.3.4}$$

Thus, by the embedding $W \hookrightarrow L^\infty(\Omega)$, it turns out that $\|\varphi_{n,0}\|_{L^\infty(\Omega)} \le C$ for a constant C that does not depend neither on n nor ε. By the property in (3.2.3), we infer that

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_{n,0})\,\mathrm{d}x \le \int_\Omega \widehat{\beta}(\varphi_{n,0})\,\mathrm{d}x \le C,$$

since $\widehat{\beta}$ is continuous from hypothesis (H4). This implies that, from now on, all the estimates previously derived are uniform with respect to both n and ε. We point out another bound: in view of (3.2.5) and (3.2.6), it is not difficult to check that

$$\|\beta_\varepsilon(\varphi_{n,0})\|_V \le C \tag{3.3.5}$$

since $\nabla\beta_\varepsilon(\varphi_{n,0}) = \beta'(J_\varepsilon(\varphi_{n,0}))J_\varepsilon'(\varphi_{n,0})\nabla\varphi_{n,0}$ a.e in Ω, β' is of class C^1 and J_ε is Lipschitz continuous with $J_\varepsilon(0) = 0$ and Lipschitz constant less than or equal to 1. Thanks to the bounds obtained in the third a priori estimate (see (3.2.30)), we note that (3.2.18) can be improved to

$$\|\langle\partial_t\varphi^n\rangle\|_{C^0([0,T])} \le C\|(\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\|_{L^\infty(H)} \le C. \tag{3.3.6}$$

Consequently, by testing equation (3.2.8c) with $v = |\Omega|^{-1}$ and making use of the estimates (3.2.30) and (3.2.31), we deduce that

$$\begin{aligned} \|\langle\mu^n\rangle\|_{C^0([0,T])} \le{}& \tau\|\langle\partial_t\varphi^n\rangle\|_{C^0([0,T])} \\ &+ \sup_{t\in[0,T]} C\left(\|\beta_\varepsilon(\varphi^n(t))\|_{L^1(\Omega)} + \|\varphi^n(t)\|_{L^1(\Omega)} + 1\right) \\ &+ \sup_{t\in[0,T]} C\left(\|\sigma^n(t)\|_{L^1(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)}\right) \le C. \end{aligned} \tag{3.3.7}$$

We now test equation (3.2.8a) with $v = \partial_t\theta^n$, equation (3.2.8b) with $v = \partial_t\mu^n$, and the time-differentiated form of equation (3.2.8c) with $v = \partial_t\varphi^n$. Summing up the resulting equalities and noting a cancellation, we obtain

$$\begin{aligned}
&\int_\Omega |\partial_t\theta^n|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\theta^n|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mu^n|^2 \,\mathrm{d}x \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tau^{1/2}\partial_t\varphi^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla(\partial_t\varphi^n)|^2 \,\mathrm{d}x + \int_\Omega \beta_\varepsilon'(\varphi^n)|\partial_t\varphi^n|^2 \,\mathrm{d}x \\
&= \int_\Omega u\partial_t\theta^n \,\mathrm{d}x + \int_\Omega (\ell - \Lambda)\partial_t\theta^n\partial_t\varphi^n + \int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\partial_t\mu^n \,\mathrm{d}x \\
&\quad - \int_\Omega \pi'(\varphi^n)|\partial_t\varphi^n|^2 \,\mathrm{d}x + \chi\int_\Omega \partial_t\sigma^n\partial_t\varphi^n \,\mathrm{d}x.
\end{aligned}$$

Now, since β_ε is monotone and Lipschitz continuous, its derivative β_ε' is nonnegative, and the corresponding term on the left-hand side can be neglected. Then, using Young's inequality and the hypothesis (H4) according to which π' is bounded, we arrive at

$$\begin{aligned}
&\int_\Omega |\partial_t\theta^n|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\theta^n|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mu^n|^2 \,\mathrm{d}x \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tau^{1/2}\partial_t\varphi^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla(\partial_t\varphi^n)|^2 \,\mathrm{d}x \\
&\le C\int_\Omega u^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\partial_t\theta^n|^2 \,\mathrm{d}x + C\int_\Omega |\partial_t\varphi^n|^2 \,\mathrm{d}x \\
&\quad + C\int_\Omega |\partial_t\sigma^n|^2 \,\mathrm{d}x + \int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\partial_t\mu^n \,\mathrm{d}x.
\end{aligned}$$

Integrating in time over the interval $(0,t)$ for $t \le T$ yields

$$\begin{aligned}
&\frac{1}{2}\left(\int_\Omega |\nabla\theta^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla\mu^n|^2 \,\mathrm{d}x + \int_\Omega |\tau^{1/2}\partial_t\varphi^n|^2 \,\mathrm{d}x\right) \\
&\quad + \frac{1}{2}\int_0^t\!\!\int_\Omega |\partial_t\theta^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla(\partial_t\varphi^n)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac{1}{2}\left(\|\theta_{0,n}\|_V^2 + \|\nabla\mu^n(0)\|_H^2 + \|\tau^{1/2}\partial_t\varphi^n(0)\|_H^2\right) \\
&\quad + C\int_0^t\!\!\int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\!\!\int_\Omega |\partial_t\varphi^n|^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_\Omega |\partial_t\sigma^n|^2 \,\mathrm{d}x \\
&\quad + \int_0^t\!\!\int_\Omega (\lambda_P\sigma^n - \lambda_A - \lambda_E\theta^n)\mathbb{h}(\varphi^n)\partial_t\mu^n \,\mathrm{d}x\,\mathrm{d}s \\
&=: D_{0,n} + C\int_0^t\!\!\int_\Omega u^2 \,\mathrm{d}x\,\mathrm{d}s + I_1 + I_2 + I_3.
\end{aligned} \tag{3.3.8}$$

Concerning the terms on the last line, we observe that the constant $D_{0,n}$ accounts for the contribution from the initial data. Our goal is to show that $D_{0,n}$ is bounded independently of n. To this end, we recall that from (3.3.6) and (3.3.7) it follows that

$$|\langle\partial_t\varphi^n\rangle(0)| + |\langle\mu^n\rangle(0)| \le C. \tag{3.3.9}$$

Next, we consider equations (3.2.8b) and (3.2.8c) at the initial time. We subtract the mean value from (3.2.8b) and test the resulting equation with $\mathcal{N}(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))$. Then, we test (3.2.8c) with $(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))$ and add the two resulting expressions. By property (3.2.20), we observe that the terms involving $\mu^n(0)$ cancel out, as do all scalar products between mean values and either $\mathcal{N}(\partial_t\varphi^n(0)-\langle\partial_t\varphi^n\rangle(0))$ or $(\partial_t\varphi^n(0)-\langle\partial_t\varphi^n\rangle(0))$. Performing an integration by parts, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\mathcal{N}(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))|^2 \,\mathrm{d}x + \tau\int_\Omega |\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0)|^2 \,\mathrm{d}x \\
&\quad = \langle(\lambda_P\sigma_{0,n} - \lambda_A - \lambda_E\theta_{0,n})\mathbb{h}(\varphi_{0,n}), \mathcal{N}(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))\rangle_V \\
&\qquad - \int_\Omega (-\Delta\varphi_{0,n} + \beta_\varepsilon(\varphi_{0,n}) + \pi(\varphi_{0,n}) - \chi\sigma_{0,n} - \Lambda\theta_{0,n})(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))\,\mathrm{d}x.
\end{aligned}$$

The last term can be rewritten, again using (3.2.20), as

$$-\int_\Omega \nabla(-\Delta\varphi_{0,n} + \beta_\varepsilon(\varphi_{0,n}) + \pi(\varphi_{0,n}) - \chi\sigma_{0,n} - \Lambda\theta_{0,n})\cdot\nabla\mathcal{N}(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))\,\mathrm{d}x.$$

Then, by applying the Poincaré and Young inequalities, we conclude that

$$\frac{1}{2}\int_\Omega |\nabla\mathcal{N}(\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0))|^2 \,\mathrm{d}x + \tau\int_\Omega |\partial_t\varphi^n(0) - \langle\partial_t\varphi^n\rangle(0)|^2 \,\mathrm{d}x \le C, \tag{3.3.10}$$

thanks to (3.3.3)–(3.3.5) and the bounds on the initial data, which are under control due to the regularity assumptions $\sigma_0 \in V$ and $\theta_0 \in V$ (cf. (3.1.21) and (3.2.7)). We point out that (3.3.10) and (3.3.9) yield in particular that $\|\partial_t\varphi^n(0)\|^2_{V'} \le C$, due to the equivalence of norms.
Finally, we take $v = \mu^n(0) - \langle\mu^n\rangle(0)$ in equation (3.2.8b) at the initial time. By carefully handling the terms and using the Poincaré and Young inequalities once more, we easily deduce that $\|\nabla\mu^n(0)\|^2_H \le C$, and hence, we ultimately obtain

$$D_{0,n} = \frac{1}{2}\left(\|\theta_{0,n}\|^2_V + \|\nabla\mu^n(0)\|^2_H + \|\tau^{1/2}\partial_t\varphi^n(0)\|^2_H\right) \le C \tag{3.3.11}$$

as desired.
The next step consists in estimating from above the last three terms on the right-hand side of (3.3.8). We deal with I_1 with Ehrling's Lemma applied to the spaces $V \hookrightarrow\hookrightarrow H \hookrightarrow V'$ and use estimates (3.2.34) and (3.2.18), as follows:

$$\begin{aligned}
I_1 &= C\int_0^t\!\!\int_\Omega |\partial_t\varphi^n|^2 \,\mathrm{d}x\,\mathrm{d}s \le C\int_0^t \|\partial_t\varphi^n - \langle\partial_t\varphi^n\rangle\|^2_H \,\mathrm{d}s + C\int_0^t |\langle\partial_t\varphi^n\rangle|^2 \,\mathrm{d}s \\
&\le \frac{1}{4}\int_0^t \|\nabla(\partial_t\varphi^n)\|^2_H \,\mathrm{d}s + C\int_0^t \|\partial_t\varphi^n - \langle\partial_t\varphi^n\rangle\|^2_{V'} \,\mathrm{d}s + C\int_0^t |\langle\partial_t\varphi^n\rangle|^2 \,\mathrm{d}s \\
&\le \frac{1}{4}\int_0^t \|\nabla(\partial_t\varphi^n)\|^2_H \,\mathrm{d}s + C.
\end{aligned} \tag{3.3.12}$$

Moreover, we have that

$$I_2 = C \int_\Omega |\partial_t \sigma^n|^2 \,\mathrm{d}x \le C$$

by (3.2.30). We handle I_3 by integrating by parts

$$\begin{aligned} I_3 = &- \int_0^t\!\!\int_\Omega (\lambda_P \partial_t \sigma^n - \lambda_E \partial_t \theta^n) \mathbb{h}(\varphi^n) \mu^n \,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t\!\!\int_\Omega (\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n) \mathbb{h}'(\varphi^n) \partial_t \varphi^n \mu^n \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_\Omega (\lambda_P \sigma^n - \lambda_A - \lambda_E \theta^n) \mathbb{h}(\varphi^n) \mu^n \,\mathrm{d}x \\ &- \int_\Omega (\lambda_P \sigma_{0,n} - \lambda_A - \lambda_E \theta_{0,n}) \mathbb{h}(\varphi_{0,n}) \mu^n(0) \,\mathrm{d}x =: I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4}, \end{aligned} \tag{3.3.13}$$

and then analyzing one by one its addends. The first one can be handled by Young's inequality, and estimates (3.2.30), (3.2.33):

$$\begin{aligned} I_{3,1} &\le C \int_0^t\!\!\int_\Omega (|\partial_t \sigma^n| + |\partial_t \theta^n|) |\mu^n| \,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{4} \int_0^t\!\!\int_\Omega |\partial_t \theta^n|^2 \,\mathrm{d}x\,\mathrm{d}s + C \left(\int_0^t\!\!\int_\Omega |\partial_t \sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\mu^n|^2 \,\mathrm{d}x\,\mathrm{d}s \right) \\ &\le \frac{1}{4} \int_0^t\!\!\int_\Omega |\partial_t \theta^n|^2 \,\mathrm{d}x\,\mathrm{d}s + C. \end{aligned} \tag{3.3.14}$$

Regarding the second one, we employ the Hölder inequality, estimates (3.2.30), (3.2.33), and the Young inequality, obtaining:

$$\begin{aligned} I_{3,2} &\le C \int_0^t\!\!\int_\Omega (|\sigma^n| + |\theta^n| + 1) |\partial_t \varphi^n| |\mu^n| \,\mathrm{d}x\,\mathrm{d}s \\ &\le C \int_0^t (\|\sigma^n\|_H + \|\theta^n\|_H + 1) \|\partial_t \varphi^n\|_{L^4(\Omega)} \|\mu^n\|_{L^4(\Omega)} \,\mathrm{d}s \\ &\le C \int_0^t \|\partial_t \varphi^n\|^2_{L^4(\Omega)} \,\mathrm{d}s + C \int_0^t \|\mu^n\|_V^2 \,\mathrm{d}s \le C \int_0^t \|\partial_t \varphi^n\|^2_{L^4(\Omega)} \,\mathrm{d}s + C. \end{aligned}$$

To conclude, we add and subtract to $\partial_t \varphi^n$ its mean value. Then, we apply Ehrling's lemma to the compact embeddings $V \hookrightarrow\hookrightarrow L^4(\Omega) \hookrightarrow V'$, similarly to what we did in (3.3.12). We deduce that

$$I_{3,2} \le C \int_0^t \|\partial_t \varphi^n\|^2_{L^4(\Omega)} \,\mathrm{d}s + C \le \frac{1}{4} \int_0^t \|\nabla(\partial_t \varphi^n)\|_H^2 \,\mathrm{d}s + C. \tag{3.3.15}$$

We turn our attention to $I_{3,3}$. Exploiting the Hölder and Poincaré inequalities, and estimates (3.2.30) and (3.3.7), we find

$$\begin{aligned} I_{3,3} &\le C \int_\Omega (|\sigma^n| + |\theta^n| + 1) |\mu^n| \,\mathrm{d}x \le C \left(\|\sigma^n\|_H + \|\theta^n\|_H + 1 \right) \|\mu^n\|_H \\ &\le C \|\mu^n\|_H \le C \|\nabla \mu^n\|_H + C |\langle \mu^n \rangle| \le \frac{1}{4} \|\nabla \mu^n\|_H^2 + C. \end{aligned} \tag{3.3.16}$$

Finally, recalling (3.3.11) and (3.3.9), it is easy to conclude that $I_{3,4} \leq C$.
Now, going back to (3.3.8) and collecting all the intermediate estimates, upon rearranging the terms and adjusting the constants, we infer that

$$\begin{aligned}\int_\Omega |\nabla\theta^n|^2 \,\mathrm{d}x + \int_\Omega |\nabla\mu^n|^2 \,\mathrm{d}x + \int_\Omega |\tau^{1/2}\partial_t\varphi^n|^2 \,\mathrm{d}x \\ + \int_0^t\!\!\int_\Omega |\partial_t\theta^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla(\partial_t\varphi^n)|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C,\end{aligned} \tag{3.3.17}$$

whence

$$\|\theta^n\|_{H^1(H)\cap L^\infty(V)} + \|\varphi^n\|_{H^1(V)\cap L^\infty(V)} + \|\tau^{1/2}\varphi^n\|_{W^{1,\infty}(H)} + \|\mu^n\|_{L^\infty(V)} \leq C. \tag{3.3.18}$$

Consequences of the fourth a priori estimate. Taking $v = -\Delta\varphi^n$ in (3.2.8c), which is admissible in our Faedo–Galerkin scheme, and exploiting integration by parts and monotonicity of β_ε, it is straightforward to deduce that

$$\int_\Omega |-\Delta\varphi^n|^2 \,\mathrm{d}x \leq \|\mu^n - \tau\partial_t\varphi^n - \pi(\varphi^n) + \chi\sigma^n + \Lambda\theta^n\|_H^2,$$

with the right-hand side that is uniformly bounded in $L^\infty(0,T)$ due to (3.3.18). Then, from (3.3.18) and elliptic regularity (see, e.g., [DL92], [Lio61]) it follows that

$$\|\varphi^n\|_{L^\infty(W)} \leq C. \tag{3.3.19}$$

A similar procedure can be applied to (3.2.8a) with the choice $v = -\Delta\theta^n$, in order to infer that

$$\|\theta^n\|_{L^2(W)} \leq C. \tag{3.3.20}$$

On the other hand, arguing as in (3.2.35) we find out that

$$\|\partial_t\varphi^n\|_{V'} \leq C\left(\|\nabla\mu^n\|_H + \|\sigma^n\|_H + \|\theta^n\|_H + 1\right),$$

which leads to the estimate

$$\|\varphi^n\|_{W^{1,\infty}(V')} \leq C. \tag{3.3.21}$$

Moreover, if the choose $v = -\Delta\mu^n$ in equation (3.2.8b), we deduce that

$$\|-\Delta\mu^n\|_H \leq C\left(\|\partial_t\varphi^n\|_H + \|\sigma^n\|_H + \|\theta^n\|_H + 1\right) \leq C$$

and (3.3.18) and standard elliptic regularity results ensure that

$$\|\mu^n\|_{L^2(W)} \leq C. \tag{3.3.22}$$

Passages to the limit. In the light of (3.3.18)–(3.3.22), we can conclude that the limit $(\theta_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ we found in the proof of the existence theorem as $n \to \infty$, enjoying the convergences (3.2.38)–(3.2.47), satisfies the additional estimate

$$\begin{aligned}\|\theta_\varepsilon\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} + \|\varphi_\varepsilon\|_{W^{1,\infty}(V')\cap H^1(V)\cap L^\infty(W)} \\ + \|\tau^{1/2}\varphi_\varepsilon\|_{W^{1,\infty}(H)} + \|\mu_\varepsilon\|_{L^\infty(V)\cap L^2(W)} \leq C\end{aligned} \tag{3.3.23}$$

which passes to the limit because it is independent of ε. Note that, by the Sobolev embedding $W \hookrightarrow L^\infty(\Omega)$, (3.3.23) entails as well that $\|\varphi_\varepsilon\|_{L^\infty(Q)} \leq C$. Moreover, by comparison in the equation (3.2.50) we find that

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(H)} \leq C. \tag{3.3.24}$$

Passing to the limit as $\varepsilon \to 0$, all the estimates are preserved and therefore satisfied by the limit $(\theta, \varphi, \mu, \sigma)$. Note that $\beta_\varepsilon(\varphi_\varepsilon) \to \beta(\varphi)$ weakly-$*$ in $L^\infty(0,T;H)$ due to (3.3.24). Finally, we observe that, since the right-hand side in the equation (cf. (1.3.1a) and (3.1.3))

$$\partial_t \theta - \Delta\theta = u - \ell\partial_t\varphi$$

lies in $L^2(0,T;L^6(\Omega))$, and θ_0 belongs to $L^\infty(\Omega)$, then it turns out that

$$\|\theta\|_{L^\infty(Q)} \leq C$$

by maximal parabolic regularity (see [LSU68, Chapter III, Theorem 7.1, p. 181]). This concludes the proof of Theorem 3.6.

3.4 Continuous dependence

In order to prove Theorem 3.8, we consider two pairs $\{(\theta_i, \varphi_i, \mu_i, \sigma_i)\}_{i=1,2}$ of strong solutions corresponding to the initial data $\{(\theta_{0,i}, \varphi_{0,i}, \sigma_{0,i})\}_{i=1,2}$ and to the assigned functions $\{u_i\}_{i=1,2}$. For convenience, in the following, we will employ the shorter notation

$$\begin{aligned} &\theta := \theta_1 - \theta_2, && \varphi := \varphi_1 - \varphi_2, && \mu := \mu_1 - \mu_2, && \sigma := \sigma_1 - \sigma_2, \\ &\theta_0 := \theta_{0,1} - \theta_{0,2}, && \varphi_0 := \varphi_{0,1} - \varphi_{0,2}, && \sigma_0 := \sigma_{0,1} - \sigma_{0,2}, && u := u_1 - u_2. \end{aligned}$$

Moreover, we recall the notation $\beta + \pi = \Psi'$ that we are going to use from now on. First of all, we observe that $(\theta, \varphi, \mu, \sigma)$ satisfies

$$\theta + \ell\varphi - \Delta(1 *_t \theta) = \theta_0 + \ell\varphi_0 + (1 *_t u), \tag{3.4.1a}$$
$$\partial_t\varphi - \Delta\mu = (\lambda_P\sigma - \lambda_E\theta)\mathbb{h}(\varphi_1) + (\lambda_P\sigma_2 - \lambda_A - \lambda_E\theta_2)(\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)), \tag{3.4.1b}$$
$$\tau\partial_t\varphi - \Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2) - \chi\sigma - \Lambda\theta = \mu, \tag{3.4.1c}$$
$$\begin{aligned} \partial_t\sigma - \Delta(\sigma - \chi\varphi) + \lambda_B\sigma = &- \lambda_C\sigma\mathbb{h}(\varphi_1) - \lambda_C\sigma_2(\mathbb{h}(\varphi_1) - \mathbb{h}(\varphi_2)) \\ &- \lambda_D\sigma\mathbb{k}(\theta_1) - \lambda_D\sigma_2(\mathbb{k}(\theta_1) - \mathbb{k}(\theta_2)), \end{aligned} \tag{3.4.1d}$$

a.e. in Q. This system is obtained straightforwardly by taking the difference of the systems (in the strong form) satisfied by $\{(\theta_i, \varphi_i, \mu_i, \sigma_i)\}_{i=1,2}$, and integrating the temperature equation in time. We multiply (3.4.1a) by $R\theta$, where R is a positive (big) constant yet to be determined, (3.4.1b) by φ, (3.4.1c) by $-\Delta\varphi$, and (3.4.1d) by σ. We sum all these equalities and integrate over Ω, finding

$$\begin{aligned}
& R\int_\Omega |\theta|^2\,\mathrm{d}x + \frac{R}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\nabla(1*_t\theta)|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\varphi|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\tau^{1/2}\nabla\varphi|^2\,\mathrm{d}x \\
& \quad + \int_\Omega |-\Delta\varphi|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |\nabla\sigma|^2\,\mathrm{d}x + \lambda_B\int_\Omega |\sigma|^2\,\mathrm{d}x \\
& = R\int_\Omega (\theta_0+\ell\varphi_0)\theta\,\mathrm{d}x + R\int_\Omega (1*_t u)\theta\,\mathrm{d}x \\
& \quad - R\ell\int_\Omega \varphi\theta\,\mathrm{d}x + \int_\Omega (\lambda_P\sigma - \lambda_E\theta)\mathbb{h}(\varphi_1)\varphi\,\mathrm{d}x \\
& \quad + \int_\Omega (\lambda_P\sigma_2 - \lambda_A - \lambda_E\theta_2)(\mathbb{h}(\varphi_1)-\mathbb{h}(\varphi_2))\varphi\,\mathrm{d}x \\
& \quad - \int_\Omega (\Psi'(\varphi_1)-\Psi'(\varphi_2))(-\Delta\varphi)\,\mathrm{d}x + 2\chi\int_\Omega \sigma(-\Delta\varphi)\,\mathrm{d}x + \Lambda\int_\Omega \theta(-\Delta\varphi)\,\mathrm{d}x \\
& \quad + \int_\Omega [-\lambda_C\sigma\mathbb{h}(\varphi_1) - \lambda_C\sigma_2(\mathbb{h}(\varphi_1)-\mathbb{h}(\varphi_2))]\,\sigma\,\mathrm{d}x \\
& \quad + \int_\Omega [-\lambda_D\sigma\mathbb{k}(\theta_1) - \lambda_D\sigma_2(\mathbb{k}(\theta_1)-\mathbb{k}(\theta_2))]\,\sigma\,\mathrm{d}x.
\end{aligned} \tag{3.4.2}$$

Then, recalling assumptions (H3), (H4), and the regularity estimate (3.1.25), we exploit the following facts:

- $\mathbb{h}$ and $\mathbb{k}$ are bounded and Lipschitz continuous;
- θ_2 is bounded in $L^\infty(Q)$;
- σ_2 is bounded in $L^2(0,T;W)$, and it holds that $\|\sigma_2\|_{L^\infty(\Omega)} \le C\|\sigma_2\|_W$ a.e. in $(0,T)$;
- Ψ' is locally Lipschitz continuous, and φ_1, φ_2 are bounded in $L^\infty(Q)$.

Using Young's inequality and the bounds above, the right-hand side of (3.4.2) can be estimated by

$$\begin{aligned}
& \frac{R}{2}\int_\Omega |\theta|^2\,\mathrm{d}x + CR\left(\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \int_\Omega |1*_t u|^2\,\mathrm{d}x + \int_\Omega |\varphi|^2\,\mathrm{d}x\right) \\
& \quad + \frac{1}{2}\int_\Omega |\theta|^2\,\mathrm{d}x + C\left(\int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |\varphi|^2\,\mathrm{d}x\right) + C\,(1+\|\sigma_2\|_W)\int_\Omega |\varphi|^2\,\mathrm{d}x \\
& \quad + 8\chi^2\int_\Omega |\sigma|^2\,\mathrm{d}x + 2\Lambda^2\int_\Omega |\theta|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |-\Delta\varphi|^2\,\mathrm{d}x \\
& \quad + C\int_\Omega |\sigma|^2\,\mathrm{d}x + C\|\sigma_2\|_W\int_\Omega |\varphi|^2 + |\sigma|^2\,\mathrm{d}x + C\|\sigma_2\|_W\int_\Omega |\theta||\sigma|\,\mathrm{d}x.
\end{aligned} \tag{3.4.3}$$

Let us briefly comment on the fact that, at this stage, the constants C used above depend on the norms of the solutions $\{(\theta_i,\varphi_i,\mu_i,\sigma_i)\}_{i=1,2}$ through estimate (3.1.25). However, as a consequence of the proof we are currently carrying out, we will establish the uniqueness of the solution. This implies that any solution must coincide with the one constructed in Theorems 3.5 and 3.6. Therefore, it satisfies the a priori estimates derived therein, which ensure that the norm of each component of the solution can be bounded by a constant depending only on the data of the problem—such as the final time T, the domain Ω, the constant R, the initial data... As a result, the same conclusion applies to the constants C appearing above. Most of the terms in the previous expression (3.4.3) do not need any further treatment because, after time integration, we are going to apply the Gronwall Lemma. Only the last one requires some additional calculations. By the Young inequality, we find

$$C\|\sigma_2\|_W \int_\Omega |\theta||\sigma| \,\mathrm{d}x \le \frac{1}{2}\int_\Omega |\theta|^2 \,\mathrm{d}x + C\|\sigma_2\|_W^2 \int_\Omega |\sigma|^2 \,\mathrm{d}x. \tag{3.4.4}$$

Now we collect the contibutions (3.4.2)–(3.4.4), fix $R > \frac{R}{2} + 1 + 2\Lambda^2$ (cf. the coefficients of the terms with θ on the right-hand side), then move to the left-hand side the terms in θ and $-\Delta\varphi$, and integrate in time over $(0,t)$, obtaining:

$$\begin{aligned}
&\int_\Omega |\nabla(1 *_t \theta)|^2 \,\mathrm{d}x + \int_\Omega |\varphi|^2 \,\mathrm{d}x + \int_\Omega |\tau^{1/2}\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega |\sigma|^2 \,\mathrm{d}x \\
&\quad + \int_0^t\!\!\int_\Omega |\theta|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |-\Delta\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla\sigma^n|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le C\bigg[\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \tau^{1/2}\|\nabla\varphi_0\|_H^2 + \|\sigma_0\|_H^2 + \int_0^t\!\!\int_\Omega |1 *_t u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + C\int_0^t (1 + \|\sigma_2\|_W) \int_\Omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \left(1 + \|\sigma_2\|_W^2\right) \int_\Omega |\sigma|^2 \,\mathrm{d}x\,\mathrm{d}s\bigg].
\end{aligned}$$

Hence, since the function $t \mapsto 1 + \|\sigma_2\|_W^2$ is bounded in $L^1(0,T)$, by the Gronwall Lemma we infer that

$$\begin{aligned}
&\|\theta\|_{L^2(H)}^2 + \|\nabla(1 *_t \theta)\|_{L^\infty(H)}^2 + \|\varphi\|_{L^\infty(H)\cap L^2(W)}^2 \\
&\quad + \|\tau^{1/2}\varphi\|_{L^\infty(V)}^2 + \|\sigma\|_{L^\infty(H)\cap L^2(V)}^2 \\
&\quad \le C\left(\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \|\tau^{1/2}\varphi_0\|_V^2 + \|\sigma_0\|_H^2 + \|1 *_t u\|_{L^2(H)}^2\right).
\end{aligned} \tag{3.4.5}$$

Finally, we want to improve this result for the temperature variable θ. To do so, we consider the equation satisfied by θ a.e. in Q, which is the following (cf. (1.3.1a))

$$\partial_t(\theta + \ell\varphi) - \Delta\theta = u.$$

We multiply it by $\theta + \ell\varphi$ and integrate over $\Omega \times (0,t)$, finding

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\theta + \ell\varphi|^2 \,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \frac{1}{2}\int_\Omega |\theta_0 + \ell\varphi_0|^2 \,\mathrm{d}x + \int_0^t\!\!\int_\Omega u(\theta + \ell\varphi) \,\mathrm{d}x\,\mathrm{d}s - \ell\int_0^t\!\!\int_\Omega \nabla\theta \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We apply the Young inequality and the previously proved inequality (3.4.5) to treat the terms on the right-hand side, leading to

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\theta + \ell\varphi|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t\!\!\int_\Omega |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leq C\biggl(\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \int_0^t\!\!\int_\Omega |u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^t\!\!\int_\Omega |\theta|^2 + |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \biggr) \\
&\quad \leq C\biggl(\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \tau^{1/2}\|\nabla\varphi_0\|_H^2 + \|\sigma_0\|_H^2 + \int_0^T\!\!\int_\Omega |u|^2 \,\mathrm{d}x\,\mathrm{d}s \biggr).
\end{aligned}$$

Then, we end up with

$$\begin{aligned}
&\|\theta + \ell\varphi\|_{L^\infty(H)} + \|\nabla\theta\|_{L^2(H)} \\
&\quad \leq C\left(\|\theta_0\|_H^2 + \|\varphi_0\|_H^2 + \|\tau^{1/2}\varphi_0\|_V^2 + \|\sigma_0\|_H^2 + \|u\|_{L^2(H)}^2 \right)
\end{aligned} \tag{3.4.6}$$

and the continuous dependence inequality (3.1.26) follows easily from estimates (3.4.5)–(3.4.6). Therefore, Theorem 3.8 is completely proved.

Chapter 4

A phase field tumor growth model with damage and mechanical effects

The first purpose of this chapter is to prove the existence of weak solutions to the initial-boundary value problem (1.3.5)–(1.3.7). To do so, we follow [Cav25], introducing an appropriate time-discretised and regularised version of the system. Then, we show that the discrete problem is well-posed and that its solution satisfies some a priori estimates. Finally, employing compactness results, we pass to the limit as the time-step tends to 0 and prove that the limit we find solves the original PDE system. The main mathematical challenges that we face are the following.

- The presence of the mass source in the Cahn–Hilliard equation (1.3.5a)–(1.3.5b), which implies that there is no mass conservation, i.e., the mean value of φ is not constant. This is expected from the modelling point of view; however, it requires handling the term
$$\int_\Omega U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z)\mu \,\mathrm{d}x$$
in the energy estimate (see the proof of Proposition 4.16).

- The nonlinear coupling between the single equations. In particular, in the damage equation (1.3.5e), the term
$$W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z) = \frac{1}{2}h'(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : (\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi)$$
is quadratic in $\varepsilon(\boldsymbol{u})$. In order to pass to the limit in this term from the discrete to the continuous problem, we have to perform a suitable regularity estimate for the displacement $\boldsymbol{u}$ to obtain strong convergence for $\varepsilon(\boldsymbol{u})$. This estimate, in turn, requires a $L^\infty(0,T;Z)$ uniform bound for the damage z, with $p > d$: although the p-Laplacian operator in (1.3.5e) is a nonlinear operator which complicates the

analysis, it has a fundamental regularising role. For the same reason, since we do not have uniform estimates for φ in equally strong spaces, we cannot allow a dependence of the elasticity tensor on the phase. In the literature (see, e.g., [Hei+17]) this issue has been addressed by considering the p-Laplacian regularisation $-\Delta_p\varphi$ instead of $-\Delta\varphi$ in equation (1.3.5b). However, we will not follow this strategy here.

- The damage equation is highly nonlinear due to the presence of $-\Delta_p z$ and the subdifferential $\beta = \partial\widehat{\beta}(z)$.

The second purpose of this chapter is to discuss a continuous dependence result and uniqueness, which are not addressed in [Cav25] and remain open problems. Notice that the $p-$Laplacian operator in the damage equation seems to affect the possibility of gaining uniqueness due to its degenerate character. As already pointed out in [RR14] for a similar equation, this difficulty may be overcome by replacing the degenerate p-Laplacian operator $-\operatorname{div}(|\nabla z|^{p-2}\nabla z)$ with the non-degenerate one $-\operatorname{div}((1+|\nabla z|^2)^{\frac{p-2}{2}}\nabla z)$ or with the fractional s-Laplacian (see, e.g., [RR14, p. 1282] for a definition). We follow the first strategy in Section 4.4, where we introduce a modified system which we prove to be well-posed. Even though it does not solve the original problem, this could be a first step in that direction.

4.1 Hypotheses

Let $d = 2, 3$ denote the space dimension and Ω a bounded C^2-domain in $\mathbb{R}^d$.

(A1) Regarding the nonlinear sources U and S defined in (1.3.8) and (1.3.9), we consider

$$\lambda_p,\ \lambda_a,\ \lambda_c \ \text{ non-negative constants}, \tag{4.1.1}$$
$$g \in C^0(\mathbb{R}^2),\ \text{ non-negative and bounded}, \tag{4.1.2}$$
$$f \in L^\infty(0,T;H), \tag{4.1.3}$$
$$\Lambda_c \in C^0(\mathbb{R}),\ \text{ non-negative and bounded}, \tag{4.1.4}$$
$$\sigma_c \in L^\infty(Q),\ \text{non-negative}. \tag{4.1.5}$$

(A2) Regarding the smooth potential $\Psi \in C^1(\mathbb{R})$, we suppose that the following growth conditions hold

$$\Psi(r) \geq C_1|r|^2 - C_2, \tag{4.1.6}$$
$$|\Psi'(r)| \leq C_3\Psi(r) + C_4 \tag{4.1.7}$$

for some fixed positive constants C_1, C_2, C_3, C_4 and for every $r \in \mathbb{R}$.
Moreover, we assume that there exists a convex-concave splitting $\Psi = \breve{\Psi} + \widehat{\Psi}$ such that

$$\breve{\Psi}, \widehat{\Psi} \in C^1(\mathbb{R}), \tag{4.1.8}$$

$$\breve{\Psi} \text{ is convex and its derivative satisfies } \breve{\Psi}'(0) = 0, \tag{4.1.9}$$

$$\widehat{\Psi} \text{ is concave,} \tag{4.1.10}$$

$$\widehat{\Psi}' \text{ is Lipschitz continuous.} \tag{4.1.11}$$

Remark 4.1. We point out that requiring the nonconvex part of the decomposition to be concave is not restrictive. In fact, for every $\breve{\Psi}, \widetilde{\Psi} \in C^1(\mathbb{R})$ such that $\Psi = \breve{\Psi} + \widetilde{\Psi}$, where $\breve{\Psi}$ satisfies (4.1.9) and $\widetilde{\Psi}'$ satisfies (4.1.11) with a Lipschitz constant L, we can consider

$$\Psi(r) = \left(\breve{\Psi}(r) + \frac{L}{2}r^2\right) + \left(\widetilde{\Psi}(r) - \frac{L}{2}r^2\right)$$

for every $r \in \mathbb{R}$, which is compliant to (4.1.8)–(4.1.11).

Remark 4.2. Note that hypothesis (A2) is compatible with the classical choice (1.2.3). However, it does not allow us to consider singular potential, such as of logarithm type (1.2.2). This means that we cannot guarantee that φ takes values in the physically relevant interval $[-1, 1]$.

(A3) We assume that the fourth-order elasticity tensor $\mathcal{C}$ in (1.3.10) belongs to the space $C^1(\Omega; \mathbb{R}^{d\times d\times d\times d})$ and is

$$\text{Lipschitz continuous and bounded,} \tag{4.1.12}$$

$$\text{symmetric (i.e., it satisfies (2.6.1)),} \tag{4.1.13}$$

$$\text{strongly elliptic (i.e., it satisfies (2.6.2)).} \tag{4.1.14}$$

Regarding the fourth-order viscous tensor $\mathcal{V}$, we suppose that it is of the form

$$\mathcal{V} = \omega\mathcal{C} \tag{4.1.15}$$

for a positive constant ω.

Remark 4.3. It is worth pointing out that the viscosity tensor is usually assumed to be only symmetric and positively defined. The stronger assumption (4.1.15) is made in order to prove the desired regularity for the displacement $\boldsymbol{u}$. Without it, our argument does not apply anymore (see the proof of Proposition 4.13 below).

(A4) We require that the scalar function h in (1.3.10) is of class $C^2(\mathbb{R})$ and that

$$h \text{ and } h' \text{ are Lipschitz continuous,} \tag{4.1.16}$$

$$h \text{ is bounded with } 0 \leq h \leq h^*. \tag{4.1.17}$$

We postulate that the viscosity coefficient a is $C^1(\mathbb{R})$ and that it satisfies

$$a \text{ is Lipschitz continuous,} \tag{4.1.18}$$

$$a \text{ is bounded with } 0 < a_* \leq a \leq a^*. \tag{4.1.19}$$

(A5) We assume that the constant p that occurs in the p-Laplacian $-\Delta_p$ in the damage equation (1.3.5e) satisfies

$$p > d \tag{4.1.20}$$

where d is the space dimension.

(A6) We consider a function $\widehat{\pi} \in C^1(\mathbb{R})$ with derivative $\pi := \widehat{\pi}'$ that satisfies

$$\pi \text{ is Lipschitz continuous.} \tag{4.1.21}$$

(A7) Let $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ be a function

$$\text{proper, convex and lower semicontinuous} \tag{4.1.22}$$

$$\text{with } \operatorname{int}(\mathcal{D}(\widehat{\beta})) \neq \emptyset, \tag{4.1.23}$$

and denote by $\beta := \partial\widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$ its subdifferential.

Remark 4.4. Note that hypothesis (A7) is quite general, and is compatible with a large class of potentials. As pointed out in Chapter 1, the simplest example is the following

$$\widehat{\beta}(r) = I_{[0,1]}(r) = \begin{cases} 0 & \text{if } r \in [0,1], \\ +\infty & \text{otherwise.} \end{cases}$$

(A8) Regarding the boundary conditions (1.3.6b) for the nutrient, we assume that

$$\sigma_\Gamma \in L^\infty(\Sigma) \text{ and } \sigma_\Gamma \geq 0, \tag{4.1.24}$$

$$\alpha \geq 0. \tag{4.1.25}$$

(A9) Regarding the initial conditions (1.3.7), we assume that

$$\varphi_0 \in V, \ \Psi(\varphi_0) \in L^1(\Omega), \tag{4.1.26}$$

$$\sigma_0 \in H, \ 0 \leq \sigma_0 \leq M := \max\{\|\sigma_c\|_{L^\infty(Q)}, \|\sigma_\Gamma\|_{L^\infty(\Sigma)}\}, \tag{4.1.27}$$

$$\boldsymbol{u}_0 \in W_0, \ \boldsymbol{v}_0 \in V_0, \tag{4.1.28}$$

$$z_0 \in \mathcal{D}(-\Delta_p), \ \widehat{\beta}(z_0) \in L^1(\Omega). \tag{4.1.29}$$

4.2 Existence of weak solutions

Definition 4.5. *We say that a quintuple* $(\varphi, \mu, \sigma, \boldsymbol{u}, z)$ *is a weak solution to the PDE system* (1.3.5)–(1.3.7) *if it has the regularity*

$$\varphi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V'), \quad \mu \in L^2(0,T;V),$$
$$\sigma \in L^2(0,T;V) \cap H^1(0,T;V'),$$
$$\boldsymbol{u} \in H^1(0,T;W_0) \cap W^{1,\infty}(0,T;V_0) \cap H^2(0,T;H),$$

$$z \in L^\infty(0,T;Z) \cap H^1(0,T;H),\ -\Delta_p z \in L^2(0,T;H)$$

and there exists a subgradient

$$\xi \in L^2(0,T;H) \text{ with } \xi \in \beta(z) \text{ a.e. in } Q,$$

such that the following equations are satisfied a.e. in $(0,T)$

$$\langle \partial_t \varphi, \zeta \rangle_V + \int_\Omega \nabla \mu \cdot \nabla \zeta \,\mathrm{d}x = \int_\Omega U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z)\, \zeta \,\mathrm{d}x, \tag{4.2.1a}$$

$$\int_\Omega \mu \zeta \,\mathrm{d}x = \int_\Omega \nabla \varphi \cdot \nabla \zeta \,\mathrm{d}x + \int_\Omega \Psi'(\varphi)\zeta \,\mathrm{d}x + \int_\Omega W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)\zeta \,\mathrm{d}x, \tag{4.2.1b}$$

$$\langle \partial_t \sigma, \zeta \rangle_V + \int_\Omega \nabla \sigma \cdot \nabla \zeta \,\mathrm{d}x + \alpha \int_\Gamma (\sigma - \sigma_\Gamma)\zeta \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega S(\varphi, \sigma, z)\zeta \,\mathrm{d}x, \tag{4.2.1c}$$

$$\int_\Omega \partial_{tt}\boldsymbol{u} \cdot \boldsymbol{\omega} \,\mathrm{d}x + \int_\Omega [a(z)\mathcal{V}\varepsilon(\partial_t \boldsymbol{u}) + W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)] : \varepsilon(\boldsymbol{\omega}) \,\mathrm{d}x = 0, \tag{4.2.1d}$$

$$\begin{aligned} &\int_\Omega \partial_t z \rho \,\mathrm{d}x + \int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla \rho \,\mathrm{d}x \\ &\quad + \int_\Omega \xi \rho \,\mathrm{d}x + \int_\Omega \pi(z)\rho \,\mathrm{d}x + \int_\Omega W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z)\rho \,\mathrm{d}x = 0 \end{aligned} \tag{4.2.1e}$$

for all $\zeta \in V$*,* $\boldsymbol{\omega} \in V_0$ *and* $\rho \in Z$*. Moreover, we require that the quintuple complies with the initial conditions, i.e.,*

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0) = \boldsymbol{v}_0, \quad z(0) = z_0 \quad \text{a.e. in } \Omega.$$

Remark 4.6. Notice that, with the regularity we demand, requiring (4.2.1b) is equivalent to asking that equation (1.3.5b) is satisfied in $L^2(0,T;H)$ and the boundary condition $\partial_{\boldsymbol{\nu}} \varphi = 0$ in (1.3.6) is satisfied in the sense of the traces. The same also holds for the damage equation. Similarly, equation (4.2.1d) is equivalent to asking that

$$\begin{aligned} &\partial_{tt}\boldsymbol{u} - a'(z)\mathcal{V}\varepsilon(\partial_t \boldsymbol{u})\nabla z - a(z)\,\mathrm{div}\,[\mathcal{V}\varepsilon(\partial_t \boldsymbol{u})] \\ &\quad - h'(z)\mathcal{C}\left(\varepsilon(\partial_t \boldsymbol{u}) - \mathcal{R}\varphi\right)\nabla z - h(z)\,\mathrm{div}\,[\mathcal{C}\varepsilon(\partial_t \boldsymbol{u}) - \mathcal{C}\mathcal{R}\varphi] = \mathbf{0} \end{aligned}$$

is satisfied in $L^2(0,T;H)$ and that the boundary condition $\boldsymbol{u} = \mathbf{0}$ in (1.3.6) holds in the sense of the traces.

Remark 4.7. Note that, by standard embedding results (see [Str66] and [LM12]),

$$\begin{aligned} \varphi &\in L^\infty(0,T;V) \cap C^0([0,T];V') &&\hookrightarrow C^0_w([0,T];V), \\ \sigma &\in L^2(0,T;V) \cap H^1(0,T;V') &&\hookrightarrow C^0([0,T];H), \\ \boldsymbol{u} &\in H^1(0,T;W_0) &&\hookrightarrow C^0([0,T];W_0), \\ \partial_t \boldsymbol{u} &\in L^\infty(0,T;V_0) \cap C^0([0,T];H) &&\hookrightarrow C^0_w([0,T];V_0), \\ z &\in L^\infty(0,T;Z) \cap C^0(0,T;H) &&\hookrightarrow C^0_w([0,T];Z), \end{aligned}$$

so $\varphi(0)$ makes sense in V, $\sigma(0)$ in H, $\boldsymbol{u}(0)$ in W_0, $\partial_t \boldsymbol{u}(0)$ in V_0 and $z(0)$ in Z. This justifies the initial data regularities that we prescribed.

Theorem 4.8. *Let hypotheses (A1)–(A9) be satisfied. Then, there exists a weak solution to system* (1.3.5)–(1.3.7) *in the sense of Definition 4.5 with the additional property that*

$$0 \le \sigma \le M \ \text{ a.e. in } Q.$$

Remark 4.9. It is not difficult to prove that if $(\varphi, \mu, \sigma, \boldsymbol{u}, z)$ is a weak solution to the PDE system (1.3.5)–(1.3.7), then φ enjoys the maximal regularity

$$\varphi \in L^4(0, T; W). \tag{4.2.2}$$

Here, we show (4.2.2) by following the approach from [GLS21b, Remark 2.3, p. 1562]. As pointed out in Remark 4.6, due to the regularity and boundary conditions fulfilled by φ, equation (1.3.5b) is satisfied in H a.e. in $(0, T)$. Explicitly, it holds

$$\int_\Omega \mu\zeta \,\mathrm{d}x = \int_\Omega \left[(-\Delta\varphi) + \Psi'(\varphi) + W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)\right] \zeta \,\mathrm{d}x$$

for every $\zeta \in H$, a.e. in $(0, T)$. Taking $\zeta = -\Delta\varphi$ as a test function, we have

$$\begin{aligned} \|-\Delta\varphi\|_H^2 &= \int_\Omega \nabla\mu \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega \Psi'(\varphi)(-\Delta\varphi) \,\mathrm{d}x - \int_\Omega W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)(-\Delta\varphi) \,\mathrm{d}x \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

where we have integrated by parts in the first term on the right-hand side and employed homogeneous Neumann boundary conditions. To handle I_1, we simply use the Hölder inequality. Regarding I_2, we write Ψ as the sum of its convex and concave parts. Then, proceeding formally, we observe that

$$\int_\Omega \breve{\Psi}'(\varphi)(-\Delta\varphi) \,\mathrm{d}x = \int_\Omega \breve{\Psi}''(\varphi)|\nabla\varphi|^2 \,\mathrm{d}x \ge 0$$

since $\breve{\Psi}$ is convex. Notice that this is not rigorous because $\breve{\Psi}$ is only C^1, but this inequality can be proved employing the Yosida–Moreau approximation of $\breve{\Psi}$, as we will do in detail in the proof of Proposition 4.16. Thus, we have

$$I_2 = -\int_\Omega [\breve{\Psi}'(\varphi) + \widehat{\Psi}'(\varphi)](-\Delta\varphi) \,\mathrm{d}x \le -\int_\Omega \widehat{\Psi}''(\varphi)|\nabla\varphi|^2 \,\mathrm{d}x \le C\|\nabla\varphi\|_H^2$$

because $\widehat{\Psi}'$ is Lipschitz continuous according to hypothesis (A2). Finally, we turn our attention to I_3. Applying the Hölder and the Young inequalities leads to

$$I_3 \le \|W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)\|_H \|-\Delta\varphi\|_H \le \frac{1}{2}\|W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)\|_H^2 + \frac{1}{2}\|-\Delta\varphi\|_H^2.$$

Then, the term related to $W_{,\varphi}$ can be treated as follows:

$$\|W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z)\|_H^2 = \int_\Omega |h(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : \mathcal{R}|^2 \,\mathrm{d}x \le C\left(\|\varepsilon(\boldsymbol{u})\|_H^2 + \|\varphi\|_H^2\right),$$

recalling that h is bounded by hypothesis (A4). Putting these estimates together, we obtain

$$\frac{1}{2}\|-\Delta\varphi\|_H^2 \leq C\left(\|\nabla\mu\|_H\|\nabla\varphi\|_H + \|\varphi\|_V^2 + \|\varepsilon(\boldsymbol{u})\|_H^2\right) \leq C\left(\|\nabla\mu\|_H + 1\right),$$

where the last inequality holds because

$$\varphi \in L^\infty(0,T;V), \qquad \boldsymbol{u} \in W^{1,\infty}(0,T;V_0).$$

Taking the square of both sides and integrating in time, we end up with

$$\|-\Delta\varphi\|_{L^4(H)}^4 \leq C\left(\|\nabla\mu\|_{L^2(H)}^2 + 1\right) \leq C,$$

since $\mu \in L^2(0,T;V)$. Thus, (4.2.2) follows from standard elliptic regularity.

4.3 Proof of Theorem 4.8

To prove the existence theorem, we will introduce a semi-implicit Euler scheme that is a time-discrete and regularised version of our system.

4.3.1 Time discretisation

Let τ be a positive and small real number. We consider a partition of $[0,T]$ with nodes

$$t_\tau^k := \begin{cases} k\tau & \text{if } k = 0,\dots,K_\tau - 1, \\ T & \text{if } k = K_\tau, \end{cases}$$

where K_τ is the ceiling integer part of T/τ, i.e., is the greatest integer such that $\tau(K_\tau - 1)$ is strictly smaller than T. We also introduce the notation:

$$I_\tau^k := \begin{cases} [0,\tau] & \text{if } k = 1, \\ (t_\tau^{k-1},\ t_\tau^k] & \text{if } k = 2,\dots,K_\tau. \end{cases}$$

With a slight abuse of terminology, we refer to the partition as uniform and to τ as its time step, even though the last interval may have a smaller length than τ. We approximate f, σ_c and σ_Γ with their local means, i.e., we define

$$f_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} f \,\mathrm{d}s, \qquad \sigma_{c,\tau}^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \sigma_c \,\mathrm{d}s, \qquad \sigma_{\Gamma,\tau}^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \sigma_\Gamma \,\mathrm{d}s,$$

for every $k = 1,\dots,K_\tau$.

Remark 4.10. It is obvious that, since $f \in L^\infty(0,T;H)$, $\sigma_c \in L^\infty(Q)$, and $\sigma_\Gamma \in L^\infty(\Sigma)$, then $f_\tau^k \in H$, $\sigma_{c,\tau}^k \in L^\infty(\Omega)$, and $\sigma_{\Gamma,\tau}^k \in L^\infty(\Gamma)$ with

$$\|f_\tau^k\|_H \leq \|f\|_{L^\infty(H)}, \ \|\sigma_{c,\tau}^k\|_{L^\infty(\Omega)} \leq \|\sigma_c\|_{L^\infty(Q)}, \ \|\sigma_{\Gamma,\tau}^k\|_{L^\infty(\Gamma)} \leq \|\sigma_\Gamma\|_{L^\infty(\Sigma)}, \tag{4.3.1}$$

for every $k = 1,\dots,K_\tau$. In addition, $0 \leq \sigma_{c,\tau}^k, \sigma_{\Gamma,\tau}^k \leq M$.

For every sequence of scalar or vector-valued functions $\{w_k\}_k$ defined over Ω, we adopt the notation:

$$D_{\tau,k}w = \frac{w_k - w_{k-1}}{\tau}, \qquad D^2_{\tau,k}w = \frac{w_k - 2w_{k-1} + w_{k-2}}{\tau^2},$$

for every k for which it makes sense. We introduce the time-discrete approximation of our problem, which is posed in Ω:

$$D_{\tau,k}\varphi - \Delta\mu_\tau^k = U_k - \tau D_{\tau,k}\mu, \tag{4.3.2a}$$

$$\mu_\tau^k = -\Delta\varphi_\tau^k + \breve{\Psi}'(\varphi_\tau^k) + \widehat{\Psi}'(\varphi_\tau^{k-1}) + W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) + \tau D_{\tau,k}\varphi, \tag{4.3.2b}$$

$$D_{\tau,k}\sigma - \Delta\sigma_\tau^k = S_k, \tag{4.3.2c}$$

$$D^2_{\tau,k}\boldsymbol{u} - \operatorname{div}\left[a(z_\tau^k)\mathcal{V}\varepsilon(D_{\tau,k}\boldsymbol{u}) + W_{,\varepsilon}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^k), z_\tau^k)\right] = \mathbf{0}, \tag{4.3.2d}$$

$$\begin{aligned} &D_{\tau,k}z - \Delta_p z_\tau^k + \beta_\tau(z_\tau^k) + \pi(z_\tau^{k-1}) \\ &\quad + \breve{W}_{3,z}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^k) + \widehat{W}_{3,z}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) = 0. \end{aligned} \tag{4.3.2e}$$

Here, for brevity, we employed the following notation for the source terms:

$$\begin{aligned} U_k &:= \left(\frac{\lambda_p\sigma_\tau^k}{1 + |W_{,\varepsilon}(\varphi_\tau^{k-1}, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})|} - \lambda_a + f_\tau^k\right) g(\varphi_\tau^{k-1}, z_\tau^{k-1}), \\ S_k &:= -\lambda_c\sigma_\tau^k g(\varphi_\tau^{k-1}, z_\tau^{k-1}) + \Lambda_c(z_\tau^{k-1})(\sigma_{c,\tau}^k - \sigma_\tau^k). \end{aligned}$$

System (4.3.2) is coupled with the boundary conditions on Γ:

$$\partial_{\boldsymbol{\nu}}\varphi_\tau^k = \partial_{\boldsymbol{\nu}}\mu_\tau^k = 0, \tag{4.3.3a}$$

$$\partial_{\boldsymbol{\nu}}\sigma_\tau^k + \alpha(\sigma_\tau^k - \sigma_{\Gamma,\tau}^k) = 0, \tag{4.3.3b}$$

$$\boldsymbol{u}_\tau^k = \mathbf{0}, \tag{4.3.3c}$$

$$(|\nabla z_\tau^k|^{p-2}\nabla z_\tau^k) \cdot \boldsymbol{\nu} = 0. \tag{4.3.3d}$$

For every $\tau > 0$ we employ a recursive procedure that, starting from the initial values

$$\varphi_\tau^0 := \varphi_0, \quad \sigma_\tau^0 := \sigma_0, \quad \boldsymbol{u}_\tau^0 := \boldsymbol{u}_0, \quad z_\tau^0 := z_0, \tag{4.3.4}$$

gives $(\varphi_\tau^k,\ \mu_\tau^k,\ \sigma_\tau^k,\ \boldsymbol{u}_\tau^k,\ z_\tau^k)$ for every $k = 1, \dots, K_\tau$ that satisfies the previous system (4.3.2)–(4.3.3) in a proper sense that will be specified in Proposition 4.13. Notice that, due to the presence of $\tau(D_{\tau,k}\mu) = \mu_\tau^k - \mu_\tau^{k-1}$ in the discrete Cahn–Hilliard equation (4.3.2a), at the step $k = 1$ the term μ_τ^0 appears. So, we define

$$\mu_\tau^0 := 0.$$

Similarly, to give a meaning to the term $D^2_{\tau,k}\boldsymbol{u}$ in the displacement equation (4.3.2d) at the step $k = 1$, we introduce

$$\boldsymbol{u}_\tau^{-1} := \boldsymbol{u}_0 - \tau\boldsymbol{v}_0,$$

where $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are, respectively, the initial displacement and the initial velocity prescribed in (1.3.7). Moreover, we will sometimes denote the time-discrete velocity at the time-step k as

$$\boldsymbol{v}_\tau^k := D_{\tau,k}\boldsymbol{u}.$$

Before stating the well-posedness result for the approximate system, let us comment briefly on our discretisation scheme.

- Regarding the discrete Cahn–Hilliard equation (4.3.2a)–(4.3.2b), we added the regularising terms $-\tau D_{\tau,k}\mu = \mu_\tau^{k-1} - \mu_\tau^k$ and $\tau D_{\tau,k}\varphi = \varphi_\tau^k - \varphi_\tau^{k-1}$ respectively to (4.3.2a) and (4.3.2b). As will be shown in the proof of Proposition 4.13, the contribution $-\tau D_{\tau,k}\mu$ allows us to rewrite the equations (4.3.2a)–(4.3.2b) in an equivalent abstract form for which the existence of a solution is automatically guaranteed. This formulation is obtained thanks to the term $(I - \Delta)\mu_\tau^k$ that appears in equation (4.3.2a). Thus, we can apply the inverse of $(I - \Delta)$ that, as we will see, has some good properties, obtain μ_τ^k and substitute it in equation (4.3.2b). On the other hand, thanks to $-\tau D_{\tau,k}\varphi$, the term φ_τ^k appears in equation (4.3.2b). It guarantees some coercivity and ensures uniqueness of the solution. Notice that both terms $D_{\tau,k}\mu$ and $D_{\tau,k}\varphi$ are multiplied by τ, so they are expected to vanish as $\tau \to 0$. The second choice we made is to evaluate $\breve{\Psi}$ at φ_τ^k and $\widehat{\Psi}$ at φ_τ^{k-1}. This is quite common and, again, motivated by some solvability issues. The main idea is to exploit the monotonicity of $\breve{\Psi}'$ to prove existence and the fact that $\widehat{\Psi}'$ is Lipschitz continuous to control the H-norm of this perturbative term.

- We employed a convex-concave splitting for W with respect to its third variable

$$\breve{W}_3(\varphi, \varepsilon(\boldsymbol{u}), z) := \frac{1}{2}\breve{h}(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : (\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi),$$
$$\widehat{W}_3(\varphi, \varepsilon(\boldsymbol{u}), z) := \frac{1}{2}\widehat{h}(z)\mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : (\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi),$$

  which, in turn, relies on a convex-concave splitting for h, given by

$$\breve{h}(z) := h(z) + \frac{1}{2}\left(\sup_{x\in\mathbb{R}} |h''(x)|\right) z^2, \qquad \widehat{h}(z) := -\frac{1}{2}\left(\sup_{x\in\mathbb{R}} |h''(x)|\right) z^2.$$

  We observe that $\breve{h}$ is convex, $\widehat{h}$ is concave, and $h = \breve{h} + \widehat{h}$. Consequently, $\breve{W}_3$ is convex, $\widehat{W}_3$ is concave, and $W = \breve{W}_3 + \widehat{W}_3$. It is worth pointing out that, since h' is Lipschitz by hypothesis (A4), the same holds for $\breve{h}'$ and $\widehat{h}'$ and, since $h > 0$, also $\breve{h} > 0$. However, $\breve{h}$ and $\widehat{h}$ are not bounded. Notice that we did not need to introduce a convex-concave decomposition for W with respect to its first and second variables because it is already convex with respect to φ and $\varepsilon(\boldsymbol{u})$. This splitting, as well as the careful choice between implicit and explicit arguments for the derivatives of W, will have a key role in carrying out the discrete energy a priori estimate in Proposition 4.16, where we will employ the following trivial result, the proof of which is just a simple application of convex and concave inequalities.

Lemma 4.11. *Let $F : \mathbb{R} \to \mathbb{R}$ be a differentiable function that admits a convex-concave decomposition $F = \breve{F} + \widehat{F}$ with differentiable $\breve{F}$ and $\widehat{F}$. Then,*

$$(\breve{F}'(x) + \widehat{F}'(y))(x - y) \geq F(x) - F(y)$$

for every $x, y \in \mathbb{R}$.

- Finally, we replaced $\widehat{\beta}$ with its Moreau–Yosida approximation $\widehat{\beta}_\tau$ defined by

$$\widehat{\beta}_\tau(z) := \min_{y\in\mathbb{R}} \left\{ \frac{1}{2\tau}|y - z|^2 + \widehat{\beta}(y) \right\} \qquad \forall z \in \mathbb{R},$$

  and, consequently, the maximal monotone operator β in the damage equation with $\beta_\tau := (\widehat{\beta}_\tau)'$. Note that we set the regularisation parameter equal to the time step τ so that we will pass to the limit simultaneously in the Yosida regularisation and in the time discretisation as $\tau \to 0$.

Remark 4.12. We recall that $\widehat{\beta}_\tau \in C^1(\mathbb{R})$ is still convex and that β_τ is non-decreasing and Lipschitz continuous with Lipschitz constant bounded by τ^{-1} (see Proposition 2.16 and Lemma 2.20). Moreover, since $\widehat{\beta}$ is non-negative, $\widehat{\beta}_\tau$ is non-negative. Finally, it is obvious by the definition of Moreau–Yosida approximation that $\widehat{\beta}_\tau(z) \leq \widehat{\beta}(z)$ for every $z \in \mathbb{R}$.

Proposition 4.13. *Let hypotheses (A1)–(A9) be satisfied. Then, for every $k = 1 \dots K_\tau$, there exists a unique weak solution*

$$(\varphi_\tau^k, \mu_\tau^k, \sigma_\tau^k, \boldsymbol{u}_\tau^k, z_\tau^k) \in W \times W \times V \times W_0 \times \mathcal{D}(-\Delta_p)$$

to system (4.3.2)–(4.3.3) in the sense that it satisfies the boundary conditions (4.3.3) in the sense of traces, equations (4.3.2a), (4.3.2b), (4.3.2d), and (4.3.2e) hold a.e. in Ω, and equation (4.3.2c) plus boundary condition (4.3.3b) hold in the weak sense

$$\int_\Omega D_{\tau,k}\sigma\zeta \,\mathrm{d}x + \int_\Omega \nabla\sigma_\tau^k \cdot \nabla\zeta \,\mathrm{d}x + \alpha \int_\Gamma (\sigma_\tau^k - \sigma_{\Gamma,\tau}^k)\zeta \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega S_k\zeta \,\mathrm{d}x$$

for all $\zeta \in V$.

Proof. ***Nutrient equation.*** First of all, we can rewrite the system

$$\begin{cases} D_{\tau,k}\sigma - \Delta\sigma_\tau^k = -\lambda_c\sigma_\tau^k g(\varphi_\tau^{k-1}, z_\tau^{k-1}) + \Lambda_c(z_\tau^{k-1})(\sigma_{c,\tau}^k - \sigma_\tau^k) & \text{in } \Omega \\ \partial_{\boldsymbol{\nu}}\sigma_\tau^k + \alpha(\sigma_\tau^k - \sigma_{\Gamma,\tau}^k) = 0 & \text{on } \Gamma \end{cases} \tag{4.3.5}$$

in the more convenient form

$$\begin{cases} -\Delta\sigma_\tau^k + c_k\sigma_\tau^k = d_k & \text{in } \Omega \\ \partial_{\boldsymbol{\nu}}\sigma_\tau^k + \alpha(\sigma_\tau^k - \sigma_{\Gamma,\tau}^k) = 0 & \text{on } \Gamma, \end{cases} \tag{4.3.6}$$

where

$$c_k := \frac{1}{\tau} + \lambda_c g(\varphi_\tau^{k-1}, z_\tau^{k-1}) + \Lambda_c(z_\tau^{k-1}), \qquad d_k = \frac{\sigma_\tau^{k-1}}{\tau} + \Lambda_c(z_\tau^{k-1})\sigma_{c,\tau}^k \tag{4.3.7}$$

are known terms in $L^\infty(\Omega)$ and H respectively, with $c_k \geq 0$ a.e. in Ω. The variational formulation of the problem is the following:

$$\begin{cases} \text{Find a } \sigma_k \in V \text{ such that } \forall \zeta \in V \\ \displaystyle\int_\Omega \nabla\sigma_\tau^k \cdot \nabla\zeta \,\mathrm{d}x + \alpha \int_\Gamma \sigma_\tau^k \zeta \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega c_k \sigma_\tau^k \zeta \,\mathrm{d}x = \int_\Omega d_k \zeta \,\mathrm{d}x + \int_\Gamma \sigma_{\Gamma,\tau}^k \zeta \,\mathrm{d}\mathcal{H}^{d-1}. \end{cases}$$

Using Lax–Milgram theorem, one can show that there exists a unique weak solution $\sigma_\tau^k \in V$.

Cahn–Hilliard equation. We consider the problem:

$$\begin{cases} D_{\tau,k}\varphi - \Delta\mu_\tau^k = U_k - \tau D_{\tau,k}\mu & \text{in } \Omega \\ \mu_\tau^k = -\Delta\varphi_\tau^k + \breve{\Psi}'(\varphi_\tau^k) + \widehat{\Psi}'(\varphi_\tau^{k-1}) + W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) + \tau D_{\tau,k}\varphi & \text{in } \Omega \\ \partial_{\boldsymbol{\nu}}\varphi_\tau^k = \partial_{\boldsymbol{\nu}}\mu_\tau^k = 0 & \text{on } \Gamma. \end{cases} \tag{4.3.8}$$

The first equation in (4.3.8) can be reformulated in the equivalent form

$$\frac{1}{\tau}(I-\Delta)^{-1}\varphi_\tau^k + \mu_\tau^k = (I-\Delta)^{-1}\left(U_k + \mu_\tau^{k-1} + \frac{1}{\tau}\varphi_\tau^{k-1}\right), \tag{4.3.9}$$

observing that $I - \Delta : \mathcal{D}(-\Delta) \subseteq H \to H$ (with Neumann homogeneous boundary condition) is a bijective operator, so $\gamma := (I-\Delta)^{-1} : H \to H$ is injective. Moreover, $-\Delta : \mathcal{D}(-\Delta) \subseteq H \to H$ is a linear single-valued maximal monotone operator and, as a consequence, γ is a linear, single-valued, monotone, and contractive operator defined on all H. Substituting μ_τ^k in the second equation of (4.3.8) and recalling the expression of $W_{,\varphi}$ from (1.3.11), we obtain:

$$\begin{aligned} &\frac{1}{\tau}\gamma(\varphi_\tau^k) - \Delta\varphi_\tau^k + \breve{\Psi}'(\varphi_\tau^k) + \left(h(z_\tau^{k-1})\mathcal{C}\mathcal{R} : \mathcal{R} + 1\right)\varphi_\tau^k \\ &\quad = \gamma\left(U_k + \mu_\tau^{k-1} + \frac{1}{\tau}\varphi_\tau^{k-1}\right) - \widehat{\Psi}'(\varphi_\tau^{k-1}) + \varphi_\tau^{k-1} + h(z_\tau^{k-1})\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^{k-1}) : \mathcal{R}. \end{aligned}$$

For brevity, we introduce the known functions

$$\begin{aligned} j_k &:= \gamma\left(U_k + \mu_\tau^{k-1} + \frac{1}{\tau}\varphi_\tau^{k-1}\right) - \widehat{\Psi}'(\varphi_\tau^{k-1}) + \varphi_\tau^{k-1} + h(z_\tau^{k-1})\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^{k-1}) : \mathcal{R}, \\ l_k &:= h(z_\tau^{k-1})\mathcal{C}\mathcal{R} : \mathcal{R} + 1. \end{aligned}$$

We notice that $j_k \in H$ and that l_k is bounded from above, since h and $\mathcal{C}$ are bounded by hypotheses (A4) and (A3) respectively, and satisfies $l_k \geq 1$, because h is non-negative and $\mathcal{C}$ is strongly elliptic by hypotheses (A4) and (A3). To find a solution for

$$\frac{1}{\tau}\gamma(\varphi_\tau^k) - \Delta\varphi_\tau^k + \breve{\Psi}'(\varphi_\tau^k) + l_k\varphi_\tau^k = j_k, \tag{4.3.10}$$

we introduce $\breve{\Psi}_\delta$, the Moreau–Yosida approximation of $\breve{\Psi}$ with regularisation parameter $\delta > 0$. We define the operator

$$B^\delta_{\tau,k} := \frac{1}{\tau}\gamma + \breve{\Psi}'_\delta + l_k I : H \to H.$$

We can reformulate the regularised system in the abstract form:

$$(B^\delta_{\tau,k} - \Delta)(\varphi_\delta) = j_k. \tag{4.3.11}$$

The operator $-\Delta$ is maximal monotone. $B^\delta_{\tau,k}$ is monotone (because it is the sum of monotone operators) and hemicontinuous (because it is continuous). Finally, it is easy to show that $B^\delta_{\tau,k} - \Delta$ is coercive. So, we can apply [Bar76, Corollary 1.3, p. 48], and conclude that $B^\delta_{\tau,k} - \Delta$ is maximal monotone and that its range is equal to H. This leads to the fact that it exists a

$$\varphi_\delta \in \mathcal{D}(B^\delta_{\tau,k} - \Delta) = H \cap \mathcal{D}(-\Delta) = W$$

that satisfies (4.3.11). Note that, obviously, φ_δ also depends on k and τ, but at this level, they are fixed, so we omit this dependence to avoid overloading the notation. Now it only remains to pass to the limit for $\delta \to 0$ and show that the limit satisfies (4.3.10). We need some a priori estimates.

First a priori estimate. We test (4.3.11) with φ_δ:

$$\int_\Omega \left(\frac{1}{\tau}\gamma(\varphi_\delta)\varphi_\delta + |\nabla\varphi_\delta|^2 + \breve{\Psi}'_\delta(\varphi_\delta)\varphi_\delta + l_k\varphi_\delta^2\right) \mathrm{d}x = \int_\Omega j_k\, \varphi_\delta \, \mathrm{d}x.$$

Using the fact that γ is monotone with $\gamma(0) = 0$, that $\breve{\Psi}'_\delta$ is monotone with $\breve{\Psi}'_\delta(0) = 0$ and $l_k \geq 1$, we have

$$\|\varphi_\delta\|_V^2 \leq C\|\varphi_\delta\|_H \|j_k\|_H,$$

from which we get $\|\varphi_\delta\|_V \leq C\|j_k\|_H = C_\tau$, where C_τ does not depend on δ.

Second a priori estimate. We test (4.3.11) with $-\Delta\varphi_\delta + \breve{\Psi}'_\delta(\varphi_\delta)$ and, since l_k is uniformly bounded from above and γ is a contraction, by the first a priori estimate we get:

$$\|-\Delta\varphi_\delta + \breve{\Psi}'_\delta(\varphi_\delta)\|_H \leq \|-\frac{1}{\tau}\gamma(\varphi_\delta) - l_k\varphi_\delta + j_k\|_H \leq C_\tau.$$

On the other hand, we have

$$\begin{aligned} \|-\Delta\varphi_\delta + \breve{\Psi}'_\delta(\varphi_\delta)\|_H^2 &= \int_\Omega |-\Delta\varphi_\delta|^2 \,\mathrm{d}x + \int_\Omega |\breve{\Psi}'_\delta(\varphi_\delta)|^2 \,\mathrm{d}x + 2\int_\Omega -\Delta\varphi_\delta \, \breve{\Psi}'_\delta(\varphi_\delta) \,\mathrm{d}x \\ &= \int_\Omega |-\Delta\varphi_\delta|^2 \,\mathrm{d}x + \int_\Omega |\breve{\Psi}'_\delta(\varphi_\delta)|^2 \,\mathrm{d}x + 2\int_\Omega \breve{\Psi}''_\delta(\varphi_\delta)|\nabla\varphi_\delta|^2 \,\mathrm{d}x \\ &\geq \|-\Delta\varphi_\delta\|_H^2 + \|\breve{\Psi}'_\delta(\varphi_\delta)\|_H^2 \end{aligned}$$

because, recalling that $\breve{\Psi}'_\delta$ is Lipschitz and non-decreasing, $\breve{\Psi}''_\delta \geq 0$ a.e.

From the first and the second a priori estimates, we get that $\|\varphi_\delta\|_W + \|\breve{\Psi}'_\delta(\varphi_\delta)\|_H \leq C_\tau$, so there exist a $\varphi_\tau^k \in W$ and a $\rho_\tau^k \in H$ such that, along a non-relabelled subsequence, $\varphi_\delta \rightharpoonup \varphi_\tau^k$ in W, $\varphi_\delta \to \varphi_\tau^k$ in H and $\breve{\Psi}'_\delta(\varphi_\delta) \rightharpoonup \rho_\tau^k$ in H. Furthermore, because of these convergences,

$$\lim_{\delta\to 0} \int_\Omega \breve{\Psi}'_\delta(\varphi_\delta)\, \varphi_\delta \,\mathrm{d}x = \int_\Omega \rho_\tau^k \,\varphi_\tau^k \,\mathrm{d}x.$$

So, thanks to Proposition 2.14, we have that $\rho_\tau^k = \breve{\Psi}'(\varphi_\tau^k)$. Pointing out that $\gamma(\varphi_\delta) \to \gamma(\varphi_\tau^k)$ in H since γ is a contraction, we can pass to the weak limit in (4.3.10) and deduce that

$$\gamma(\varphi_\tau^k) - \Delta\varphi_\tau^k + \breve{\Psi}'(\varphi_\tau^k) + l_k\varphi_\tau^k = j_k$$

in H. Additionally, we remark that $\partial_{\boldsymbol{\nu}}\varphi_\tau^k = 0$ on Γ in the sense of because $\partial_{\boldsymbol{\nu}}\varphi_\delta = 0$ for every δ and the normal trace operator is linear and continuous over $H^2(\Omega)$.
Finally, we define μ_τ^k as in the second equation of system (4.3.8) and claim that it belongs to the range of $(I-\Delta)^{-1}$, i.e.,

$$\mathcal{D}(I-\Delta) = W$$

by comparison in (4.3.9). It remains to prove that the solution $(\varphi_\tau^k, \mu_\tau^k)$ is unique. We take two solutions and the components φ_1 and φ_2 solving (4.3.10). They satisfy

$$(\gamma(\varphi_1) - \gamma(\varphi_2)) - \Delta(\varphi_1 - \varphi_2) + \breve{\Psi}'(\varphi_1) - \breve{\Psi}'(\varphi_2) + l_k(\varphi_1 - \varphi_2) = 0$$

in H. Testing this equation with $\varphi_1 - \varphi_2$, we have

$$\int_\Omega (\gamma(\varphi_1) - \gamma(\varphi_2))(\varphi_1 - \varphi_2)\,\mathrm{d}x + \int_\Omega |\nabla(\varphi_1 - \varphi_2)|^2\,\mathrm{d}x \\ + \int_\Omega (\breve{\Psi}'(\varphi_1) - \breve{\Psi}'(\varphi_2))(\varphi_1 - \varphi_2)\,\mathrm{d}x + \int_\Omega l_k(\varphi_1 - \varphi_2)^2\,\mathrm{d}x = 0.$$

Since γ and $\breve{\Psi}'$ are monotone, the first and third addends are non-negative. Moreover, given that $l_k \geq 1$, we get

$$\int_\Omega (\varphi_1 - \varphi_2)^2\,\mathrm{d}x + \int_\Omega |\nabla(\varphi_1 - \varphi_2)|^2\,\mathrm{d}x \leq 0,$$

from which $\varphi_1 = \varphi_2$ follows. Consequently, also the components μ_1 and μ_2 must coincide from (4.3.9).

Damage differential equation. We want to find a weak solution of

$$\begin{cases} D_{\tau,k}z - \Delta_p z_\tau^k + \beta_\tau(z_\tau^k) + \pi(z_\tau^{k-1}) \\ \quad + \dfrac{\breve{h}'(z_\tau^k) + \widehat{h}'(z_\tau^{k-1})}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] = 0 & \text{in } \Omega \\ (|\nabla z_\tau^k|^{p-2}\nabla z_\tau^k)\cdot\boldsymbol{\nu} = 0 & \text{on } \Gamma \end{cases} \tag{4.3.12}$$

using a minimizing procedure. So we introduce the functional $\mathcal{F}_{\tau,k} : Z \to \mathbb{R}$ defined as follows:

$$\begin{aligned}\mathcal{F}_{\tau,k}(z) :=& \frac{1}{2\tau}\int_\Omega |z|^2 \,\mathrm{d}x - \frac{1}{\tau}\int_\Omega z_\tau^{k-1} z \,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla z|^p \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\tau(z)\,\mathrm{d}x \\ &+ \int_\Omega \pi(z_\tau^{k-1}) z \,\mathrm{d}x + \int_\Omega \frac{\breve{h}(z)}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k]\,\mathrm{d}x \\ &+ \int_\Omega \frac{\widehat{h}'(z_\tau^{k-1})}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] z \,\mathrm{d}x\end{aligned}$$

and we use the direct method of the Calculus of Variations. We consider a minimizing sequence $\{z_j\}_j$ and prove that it admits a subsequence that converges to a minimizer for $\mathcal{F}_{\tau,k}$. We will need coercivity and weakly lower semicontinuity of $\mathcal{F}_{\tau,k}$.

Coercivity. Recalling that $\widehat{\beta}_\tau$ and $\breve{h}$ are nonnegative, π and $\widehat{h}'$ are Lipschitz, $z_\tau^{k-1} \in Z \hookrightarrow L^\infty$ since p is strictly bigger than d, and $\mathcal{C}$ is bounded and strongly elliptic, we obtain:

$$\mathcal{F}_{\tau,k}(z) \geq \frac{1}{2\tau}\int_\Omega |z|^2 \,\mathrm{d}x - C\int_\Omega |z| \,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla z|^p \,\mathrm{d}x - C\int_\Omega |\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k|^2 |z| \,\mathrm{d}x.$$

Using the Young inequality and $\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k \in V \hookrightarrow L^4(\Omega)$, the previous inequality becomes:

$$\mathcal{F}_{\tau,k}(z) \geq C\int_\Omega |z|^2 \,\mathrm{d}x + C\int_\Omega |\nabla z|^p \,\mathrm{d}x - C.$$

Weak lower semicontinuity. All terms are convex and continuous in the strong topology and, therefore, weakly lower semicontinuous (see [Bré11, Corollary 3.9, p. 61]).

We note that it exists $C \in \mathbb{R}$ such that $\inf_Z \mathcal{F}_{\tau,k} < C$, so we can suppose without loss of generality that $\mathcal{F}_{\tau,k}(z_j) < C$ for every j. Thanks to coercivity, it trivially follows that $\{z_j\}_j$ is bounded in Z. Thus, there exists a subsequence that we do not relabel and a $z_\tau^k \in Z$ such that $z_j \rightharpoonup z_\tau^k$ in Z. From weakly lower semicontinuity, we get that:

$$\mathcal{F}_{\tau,k}(z_\tau^k) \leq \liminf_{j\to+\infty} \mathcal{F}_{\tau,k}(z_j) = \inf_Z \mathcal{F}_{\tau,k},$$

so z_τ^k is a minimizer for $\mathcal{F}_{\tau,k}$. To conclude, we observe that $\mathcal{F}_{\tau,k}$ is Fréchet differentiable, so the minimum z_τ^k satisfies the following associated Euler-Lagrange equation

$$\begin{aligned}0 =& \frac{1}{\tau}\int_\Omega z_\tau^k w \,\mathrm{d}x - \frac{1}{\tau}\int_\Omega z_\tau^{k-1} w \,\mathrm{d}x \\ &+ \int_\Omega |\nabla z_\tau^k|^{p-2}\nabla z_\tau^k \cdot \nabla w \,\mathrm{d}x + \int_\Omega \beta_\tau(z_\tau^k) w \,\mathrm{d}x + \int_\Omega \pi(z_\tau^{k-1}) w \,\mathrm{d}x \\ &+ \int_\Omega \frac{\breve{h}'(z_\tau^k) + \widehat{h}'(z_\tau^{k-1})}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] w \,\mathrm{d}x\end{aligned} \tag{4.3.13}$$

for every $w \in Z$. By comparison in (4.3.13), since

$$\frac{z_\tau^k - z_\tau^{k-1}}{\tau} + \beta_\tau(z) + \pi(z_\tau^{k-1}) + \frac{\breve{h}'(z_\tau^k) + \widehat{h}'(z_\tau^{k-1})}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k]$$

belongs to H, it follows that also $-\Delta_p z_\tau^k = -\operatorname{div}(|\nabla z_\tau^k|^{p-2}\nabla z_\tau^k)$ belongs to H. This means that $z_\tau^k \in \mathcal{D}(-\Delta_p)$. Now we prove that the solution is unique. If we suppose to have two solutions to (4.3.12) z_1 and z_2, both of them are minimizers of $\mathcal{F}_{\tau,k}$ and satisfy (4.3.13). If we consider the difference between the two equations and we take $w = z_1 - z_2$ as a test function, we obtain:

$$\begin{aligned} 0 =& \frac{1}{\tau}\int_\Omega (z_1 - z_2)^2 \,\mathrm{d}x + \int_\Omega (|\nabla z_1|^{p-2}\nabla z_1 - |\nabla z_2|^{p-2}\nabla z_2)\cdot\nabla(z_1 - z_2)\,\mathrm{d}x \\ &+ \int_\Omega \frac{\breve{h}'(z_1) - \breve{h}'(z_2)}{2}(z_1 - z_2)\mathcal{C}[\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k] : [\varepsilon(\boldsymbol{u}_\tau^{k-1}) - \mathcal{R}\varphi_\tau^k]\,\mathrm{d}x \\ &+ \int_\Omega (\beta_\tau(z_1) - \beta_\tau(z_2))\,(z_1 - z_2)\,\mathrm{d}x \geq \frac{1}{\tau}\|z_1 - z_2\|_H^2, \end{aligned}$$

where the last inequality follows from the fact that $-\Delta_p$, β_τ, and $\breve{h}'$ are monotone operators, so the related terms are non-negative. Thus, it turns out that $z_1 = z_2$.

Displacement equation. First of all, we rewrite the system

$$\begin{cases} D_{\tau,k}^2\boldsymbol{u} - \operatorname{div}\left[a(z_\tau^k)\mathcal{V}\varepsilon(D_{\tau,k}\boldsymbol{u}) + h(z_\tau^k)\mathcal{C}(\varepsilon(\boldsymbol{u}_\tau^k) - \mathcal{R}\varphi_\tau^k)\right] = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{u}_\tau^k = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{4.3.14}$$

as

$$\begin{cases} -\operatorname{div}\left[\mathcal{T}_k\varepsilon(\boldsymbol{u}_\tau^k)\right] + \boldsymbol{u}_\tau^k = \mathbf{t}_k & \text{in } \Omega \\ \boldsymbol{u}_\tau^k = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{4.3.15}$$

where we have introduced the following known terms:

$$\mathcal{T}_k := \tau a(z_\tau^k)\mathcal{V} + \tau^2 h(z_\tau^k)\mathcal{C} = (\tau\omega a(z_\tau^k) + \tau^2 h(z_\tau^k))\mathcal{C} = \theta(z_\tau^k)\mathcal{C}, \tag{4.3.16}$$

$$\mathbf{t}_k := 2\boldsymbol{u}_\tau^{k-1} - \boldsymbol{u}_\tau^{k-2} - \operatorname{div}\left[\tau^2 h(z_\tau^k)\varphi_\tau^k\mathcal{C}\mathcal{R} + \tau a(z_\tau^k)\mathcal{V}\varepsilon(\boldsymbol{u}_\tau^{k-1})\right]. \tag{4.3.17}$$

Since $\mathcal{T}_k$ is bounded and coercive and $\mathbf{t}_k$ is in H, it is easy to prove using Lax–Milgram theorem that system (4.3.15) has a (unique) weak solution $\boldsymbol{u}_\tau^k \in V_0$. It remains to be proved that $\boldsymbol{u}_\tau^k \in W_0$, and it can be done exactly as in [HR15, Lemma 4.1, p. 4596], using a bootstrap argument. Notice that this is the spot where we need to require $\mathcal{V} = \omega\mathcal{C}$.

□

Given a sequence of scalar or vector-valued functions $\{w_\tau^k\}_{k=0}^{K_\tau}$ defined over Ω, we introduce the piecewise constant interpolations $\overline{w}_\tau, \underline{w}_\tau$ and the piecewise linear interpolation w_τ over the time interval $[0, T]$ as

$$\overline{w}_\tau(t) := w_\tau^k, \qquad \underline{w}_\tau(t) := w_\tau^{k-1}, \qquad w_\tau(t) := \frac{t - t_\tau^{k-1}}{\tau}w_\tau^k + \frac{t_\tau^k - t}{\tau}w_\tau^{k-1} \tag{4.3.18}$$

for every $t \in I_\tau^k$. With this new notation, the time-discretised and regularised system (4.3.2) can be written as

$$\partial_t \varphi_\tau - \Delta \overline{\mu}_\tau = \left(\frac{\lambda_p \overline{\sigma}_\tau}{1 + |W_{,\varepsilon}(\underline{\varphi}_\tau, \varepsilon(\underline{\boldsymbol{u}}_\tau), \underline{z}_\tau)|} - \lambda_a + \overline{f}_\tau \right) g(\underline{\varphi}_\tau, \underline{z}_\tau) - (\overline{\mu}_\tau - \underline{\mu}_\tau), \quad (4.3.19a)$$

$$\overline{\mu}_\tau = -\Delta \overline{\varphi}_\tau + \breve{\Psi}'(\overline{\varphi}_\tau) + \widehat{\Psi}'(\underline{\varphi}_\tau) - h(\underline{z}_\tau)(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) : \mathcal{C}\mathcal{R} + (\overline{\varphi}_\tau - \underline{\varphi}_\tau), \quad (4.3.19b)$$

$$\partial_t \sigma_\tau - \Delta \overline{\sigma}_\tau = -\lambda_c \overline{\sigma}_\tau g(\underline{\varphi}_\tau, \underline{z}_\tau) + \Lambda_c(\underline{z}_\tau)(\overline{\sigma}_{c,\tau} - \overline{\sigma}_\tau), \quad (4.3.19c)$$

$$\partial_t \boldsymbol{v}_\tau - \operatorname{div}\left[a(\overline{z}_\tau)\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau) + h(\overline{z}_\tau)\mathcal{C}(\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau)\right] = \boldsymbol{0}, \quad (4.3.19d)$$

$$\begin{aligned} &\partial_t z_\tau - \Delta_p \overline{z}_\tau + \beta_\tau(\overline{z}_\tau) + \pi(\underline{z}_\tau) \\ &\quad + \frac{\breve{h}'(\overline{z}_\tau) + \widehat{h}'(\underline{z}_\tau)}{2}(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) : \mathcal{C}(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) = 0. \end{aligned} \quad (4.3.19e)$$

4.3.2 A priori estimates for the time-discrete system

In the following, we will need the boundedness of the nutrient variable σ_τ^k, so we prove a comparison principle.

Lemma 4.14. *The function σ_τ^k satisfies $0 \leq \sigma_\tau^k \leq M$ for every $k = 0, \dots, K_\tau$.*

Proof. Knowing that $\sigma_\tau^0 = \sigma_0$ satisfies this property by hypothesis (A9), we proceed by induction on k, so we suppose that $0 \leq \sigma_\tau^{k-1} \leq M$ and we prove that the same holds for σ_τ^k. We recall that $0 \leq \sigma_{c,\tau}^k,\ \sigma_{\Gamma,\tau}^k \leq M$ and that, using the notation introduced in (4.3.7), $c_k \geq 1/\tau$ and $d_k \geq 0$. We also recall that, given a function F, its positive and negative parts are defined as

$$F_+(x) := \max\{F(x), 0\}, \qquad F_-(x) := \max\{-F(x), 0\},$$

and that, if $F \in V$, the following relations hold

$$\begin{aligned} \int_\Omega F\, F_+ \,\mathrm{d}x &= \|F_+\|_H^2, & \int_\Omega F\, F_- \,\mathrm{d}x &= -\|F_-\|_H^2, \\ \int_\Omega \nabla F \cdot \nabla F_+ \,\mathrm{d}x &= \|\nabla F_+\|_H^2, & \int_\Omega \nabla F \cdot \nabla F_- \,\mathrm{d}x &= -\|\nabla F_-\|_H^2. \end{aligned} \quad (4.3.20)$$

Testing (4.3.2c) with $-(\sigma_\tau^k)_-$, we obtain

$$\begin{aligned} &-\int_\Omega \nabla \sigma_\tau^k \cdot \nabla[(\sigma_\tau^k)_-] \,\mathrm{d}x - \int_\Gamma \alpha(\sigma_\tau^k - \sigma_{\Gamma,\tau}^k)(\sigma_\tau^k)_- \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega c_k \sigma_\tau^k (\sigma_\tau^k)_- \,\mathrm{d}x \\ &\quad = -\int_\Omega d_k (\sigma_\tau^k)_- \,\mathrm{d}x. \end{aligned}$$

Using (4.3.20), it holds

$$\begin{aligned} \frac{1}{\tau}\|(\sigma_\tau^k)_-\|_H^2 &\leq \|\nabla[(\sigma_\tau^k)_-]\|_H^2 + \|\sqrt{c_k}(\sigma_\tau^k)_-\|_H^2 + \|\sqrt{\alpha}(\sigma_\tau^k)_-\|_{L^2_\Gamma}^2 \\ &= -\int_\Omega d_k (\sigma_\tau^k)_- \,\mathrm{d}x - \int_\Gamma \alpha\, \sigma_{\Gamma,\tau}^k (\sigma_\tau^k)_- \,\mathrm{d}\mathcal{H}^{d-1} \leq 0, \end{aligned}$$

so $\|(\sigma_\tau^k)_-\|_H^2 = 0$ (or, equivalently, $\sigma_\tau^k \geq 0$ a.e. in Ω).
In the same way, we test (4.3.2c) with $(\sigma_\tau^k - M)_+$, obtaining

$$\int_\Omega \nabla\sigma_\tau^k \cdot \nabla[(\sigma_\tau^k - M)_+]\,\mathrm{d}x + \int_\Gamma \alpha(\sigma_\tau^k - \sigma_{\Gamma,\tau}^k)(\sigma_\tau^k - M)_+\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega c_k\sigma_\tau^k(\sigma_\tau^k - M)_+\,\mathrm{d}x = \int_\Omega d_k(\sigma_\tau^k - M)_+\,\mathrm{d}x,$$

that can be rewritten as

$$\int_\Omega |\nabla[(\sigma_\tau^k - M)_+]|^2\,\mathrm{d}x + \int_\Gamma \alpha[(\sigma_\tau^k - M)_+]^2\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Gamma \alpha(\sigma_{\Gamma,\tau}^k - M)(\sigma_\tau^k - M)_+\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega c_k[(\sigma_\tau^k - M)_+]^2\,\mathrm{d}x + \int_\Omega (c_kM - d_k)(\sigma_\tau^k - M)_+\,\mathrm{d}x = 0.$$

Noticing that

$$c_kM - d_k = \frac{1}{\tau}(M - \sigma_\tau^{k-1}) + \Lambda_c(z_\tau^{k-1})(M - \sigma_{c,\tau}^k) + \lambda_c M g(\varphi_\tau^{k-1}, z_\tau^{k-1}) \geq 0,$$

and recalling that $c_k \geq 1/\tau$, from the previous inequality it follows that $\|(\sigma_\tau^k - M)_+\|_H^2$ is equal to 0, so $\sigma_\tau^k \leq M$ a.e. in Ω.

□

Remark 4.15. From Lemma 4.14 and (4.3.1) it follows that:

- $U_k = \left(\dfrac{\lambda_p\sigma_\tau^k}{1 + |W_{,\varepsilon}(\varphi_\tau^{k-1}, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})|} - \lambda_a + f_\tau^k\right) g(\varphi_\tau^{k-1}, z_\tau^{k-1})$ belongs to H with $\|U_k\|_H \leq C$ for a positive C independent of τ and k,
- $S_k = -\lambda_c\sigma_\tau^k g(\varphi_\tau^{k-1}, z_\tau^{k-1}) + \Lambda_c(z_\tau^{k-1})(\sigma_{c,\tau}^k - \sigma_\tau^k)$ is in $L^\infty(\Omega)$ with $\|S_k\|_{L^\infty} \leq C$ for a positive C independent of τ and k.

Proposition 4.16. *The time-discrete solution to the problem (4.3.19) constructed from Proposition 4.13 satisfies the following a priori estimates uniformly in τ :*

$$\|\overline{\varphi}_\tau\|_{L^\infty(V)\cap L^2(W)} + \|\underline{\varphi}_\tau\|_{L^\infty(V)} \leq C, \tag{4.3.21}$$
$$\tau^{-1/2}\|\overline{\varphi}_\tau - \underline{\varphi}_\tau\|_{L^2(H)} \leq C, \tag{4.3.22}$$
$$\|\partial_t\varphi_\tau\|_{L^2(V')} \leq C, \tag{4.3.23}$$
$$\|\breve{\Psi}'(\overline{\varphi}_\tau)\|_{L^2(H)} + \|\breve{\Psi}'(\underline{\varphi}_\tau)\|_{L^2(H)} \leq C, \tag{4.3.24}$$
$$\|\overline{\mu}_\tau\|_{L^2(V)} + \|\underline{\mu}_\tau\|_{L^2(V)} \leq C, \tag{4.3.25}$$
$$\|\overline{\sigma}_\tau\|_{L^\infty(H)\cap L^2(V)} + \|\underline{\sigma}_\tau\|_{L^\infty(H)\cap L^2(V)} \leq C, \tag{4.3.26}$$
$$\|\partial_t\sigma_\tau\|_{L^2(V')} \leq C, \tag{4.3.27}$$
$$\|\overline{\boldsymbol{u}}_\tau\|_{L^\infty(W_0)} + \|\underline{\boldsymbol{u}}_\tau\|_{L^\infty(W_0)} \leq C, \tag{4.3.28}$$
$$\|\boldsymbol{u}_\tau\|_{W^{1,\infty}(V)\cap H^1(W_0)} \leq C, \tag{4.3.29}$$

$$\|\overline{\boldsymbol{v}}_\tau\|_{L^\infty(V)\cap L^2(W_0)} + \|\underline{\boldsymbol{v}}_\tau\|_{L^\infty(V)} \leq C, \tag{4.3.30}$$

$$\|\boldsymbol{v}_\tau\|_{L^\infty(V)\cap H^1(H)} \leq C, \tag{4.3.31}$$

$$\|\overline{z}_\tau\|_{L^\infty(Z)\cap L^2(W^{1+\delta,p}(\Omega))} + \|\underline{z}_\tau\|_{L^\infty(Z)\cap L^2(W^{1+\delta,p}(\Omega))} \leq C, \tag{4.3.32}$$

$$\|z_\tau\|_{L^\infty(Z)\cap L^2(W^{1+\delta,p}(\Omega))\cap H^1(H)} \leq C, \tag{4.3.33}$$

$$\|-\Delta_p \overline{z}_\tau\|_{L^2(H)} + \|-\Delta_p \underline{z}_\tau\|_{L^2(H)} \leq C, \tag{4.3.34}$$

$$\|\beta_\tau(\overline{z}_\tau)\|_{L^2(H)} \leq C, \tag{4.3.35}$$

where $\delta \in (0, 1/p)$.

Notice that in equations (4.3.21) and (4.3.30) the estimates for the retarded piecewise constant interpolants hold in weaker spaces because they are equal to the initial data in $[0, \tau]$ and the initial data are less regular than the corresponding discrete solutions at the step $k = 1, \dots, K_\tau$.

Proof. ***Energy estimate.*** Testing (4.3.2a) with $\tau\mu_\tau^k$, we obtain:

$$\tau \int_\Omega D_{\tau,k}\varphi\, \mu_\tau^k \,\mathrm{d}x + \tau \int_\Omega |\nabla \mu_\tau^k|^2 \,\mathrm{d}x = \tau \int_\Omega U_k\, \mu_\tau^k \,\mathrm{d}x - \tau \int_\Omega (\mu_\tau^k - \mu_\tau^{k-1})\mu_\tau^k \,\mathrm{d}x.$$

Using the Young inequality to handle the last term, we have:

$$\begin{aligned} &\tau \int_\Omega D_{\tau,k}\varphi\, \mu_\tau^k \,\mathrm{d}x + \tau \int_\Omega |\nabla \mu_\tau^k|^2 \,\mathrm{d}x + \frac{\tau}{2} \int_\Omega |\mu_\tau^k|^2 \,\mathrm{d}x - \frac{\tau}{2} \int_\Omega |\mu_\tau^{k-1}|^2 \,\mathrm{d}x \\ &\leq \tau \int_\Omega U_k\, \mu_\tau^k \,\mathrm{d}x. \end{aligned} \tag{4.3.36}$$

Testing (4.3.2b) with $-(\varphi_\tau^k - \varphi_\tau^{k-1})$,

$$\begin{aligned} &- \tau \int_\Omega \mu_\tau^k\, D_{\tau,k}\varphi \,\mathrm{d}x + \int_\Omega \nabla \varphi_\tau^k \cdot \nabla (\varphi_\tau^k - \varphi_\tau^{k-1}) \,\mathrm{d}x \\ &+ \int_\Omega \Big[\breve{\Psi}'(\varphi_\tau^k) + \widehat{\Psi}'(\varphi_\tau^{k-1})\Big](\varphi_\tau^k - \varphi_\tau^{k-1}) \,\mathrm{d}x \\ &+ \int_\Omega W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})(\varphi_\tau^k - \varphi_\tau^{k-1}) \,\mathrm{d}x + \int_\Omega |\varphi_\tau^k - \varphi_\tau^{k-1}|^2 \,\mathrm{d}x = 0. \end{aligned}$$

Employing the Young inequality for the second term and Lemma 4.11 for $\Psi = \breve{\Psi} + \widehat{\Psi}$, we get

$$\begin{aligned} &- \tau \int_\Omega \mu_\tau^k\, D_{\tau,k}\varphi \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla \varphi_\tau^k|^2 \,\mathrm{d}x - \frac{1}{2} \int_\Omega |\nabla \varphi_\tau^{k-1}|^2 \,\mathrm{d}x \\ &+ \int_\Omega \Psi(\varphi_\tau^k) \,\mathrm{d}x - \int_\Omega \Psi(\varphi_\tau^{k-1}) \,\mathrm{d}x \\ &+ \int_\Omega W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})(\varphi_\tau^k - \varphi_\tau^{k-1}) \,\mathrm{d}x + \int_\Omega |\varphi_\tau^k - \varphi_\tau^{k-1}|^2 \,\mathrm{d}x \leq 0. \end{aligned} \tag{4.3.37}$$

Testing (4.3.2c) with $\tau\sigma_\tau^k$ and applying the Young inequality for the first term, we obtain:

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\sigma_\tau^k|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\sigma_\tau^{k-1}|^2\,\mathrm{d}x + \tau\int_\Omega |\nabla\sigma_\tau^k|^2\,\mathrm{d}x + \tau\int_\Gamma \alpha|\sigma_\tau^k|^2\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad\le \tau\int_\Omega S_k\,\sigma_\tau^k\,\mathrm{d}x + \tau\int_\Gamma \alpha\sigma_{\Gamma,\tau}^k\,\sigma_\tau^k\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{4.3.38}$$

Testing (4.3.2d) with $\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1} = \tau\boldsymbol{v}_\tau^k$, we get:

$$\begin{aligned}&\int_\Omega (\boldsymbol{v}_\tau^k - \boldsymbol{v}_\tau^{k-1})\cdot\boldsymbol{v}_\tau^k\,\mathrm{d}x \;+\; \tau\int_\Omega a(z_\tau^k)\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k):\varepsilon(\boldsymbol{v}_\tau^k)\,\mathrm{d}x\\ &\quad+\int_\Omega W_{,\varepsilon}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^k),z_\tau^k):(\varepsilon(\boldsymbol{u}_\tau^k)-\varepsilon(\boldsymbol{u}_\tau^{k-1}))\,\mathrm{d}x = 0.\end{aligned}$$

Exploiting the Young inequality for the first term, the fact that $a_* \le a$ and that $\mathcal{V}$ is uniformly elliptic for the second term, we have:

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\boldsymbol{v}_\tau^k|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\boldsymbol{v}_\tau^{k-1}|^2\,\mathrm{d}x + C\tau\int_\Omega |\varepsilon(\boldsymbol{v}_\tau^k)|^2\,\mathrm{d}x\\ &\quad+\int_\Omega W_{,\varepsilon}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^k),z_\tau^k):(\varepsilon(\boldsymbol{u}_\tau^k)-\varepsilon(\boldsymbol{u}_\tau^{k-1}))\,\mathrm{d}x \le 0.\end{aligned} \tag{4.3.39}$$

Finally, we test (4.3.2e) with $z_\tau^k - z_\tau^{k-1}$, obtaining:

$$\begin{aligned}&\tau\int_\Omega |D_{\tau,k}z|^2\,\mathrm{d}x + \int_\Omega |\nabla z_\tau^k|^{p-2}\nabla z_\tau^k\cdot\nabla(z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x\\ &\quad+\int_\Omega \beta_\tau(z_\tau^k)(z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x + \int_\Omega \pi(z_\tau^{k-1})(z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x\\ &\quad+\int_\Omega \Big[\breve{W}_{3,z}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^{k-1}),z_\tau^k) + \widehat{W}_{3,z}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^{k-1}),\,z_\tau^{k-1})\Big](z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x = 0.\end{aligned}$$

Employing the Young inequality for the second term, the convexity of $\widehat{\beta}_\tau$ for the third, and moving the term with π to the right-hand side, we get:

$$\begin{aligned}&\tau\int_\Omega |D_{\tau,k}z|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla z_\tau^k|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla z_\tau^{k-1}|^p\,\mathrm{d}x\\ &\quad+\int_\Omega \widehat{\beta}_\tau(z_\tau^k)\,\mathrm{d}x - \int_\Omega \widehat{\beta}_\tau(z_\tau^{k-1})\,\mathrm{d}x\\ &\quad+\int_\Omega \Big[\breve{W}_{3,z}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^{k-1}),z_\tau^k) + \widehat{W}_{3,z}(\varphi_\tau^k,\varepsilon(\boldsymbol{u}_\tau^{k-1}),\,z_\tau^{k-1})\Big](z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x\\ &\quad\le -\int_\Omega \pi(z_\tau^{k-1})(z_\tau^k - z_\tau^{k-1})\,\mathrm{d}x = -\tau\int_\Omega \pi(z_\tau^{k-1})D_{\tau,k}z\,\mathrm{d}x.\end{aligned} \tag{4.3.40}$$

Now we notice that, since W is convex with respect to its first and second variables, and since we can apply Lemma 4.11 to $W = \breve{W}_3 + \widehat{W}_3$, we have:

$$
\begin{aligned}
&W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})(\varphi_\tau^k - \varphi_\tau^{k-1}) \ge W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) - W(\varphi_\tau^{k-1}, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}),\\
&W_{,\varepsilon}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^k), z_\tau^{k-1}) : (\varepsilon(\boldsymbol{u}_\tau^k) - \varepsilon(\boldsymbol{u}_\tau^{k-1})) \ge W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^k), z_\tau^k) - W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^k),\\
&\Big[\breve{W}_{3,z}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^k) + \widehat{W}_{3,z}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1})\Big](z_\tau^k - z_\tau^{k-1})\\
&\quad \ge W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^k) - W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}).
\end{aligned}
$$

So, by summing the three above inequalities, we obtain that the left-hand side is greater than or equal to

$$
W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^k), z_\tau^k) - W(\varphi_\tau^{k-1}, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}).
$$

Adding (4.3.36), (4.3.37), (4.3.38), (4.3.39), (4.3.40) and employing the previous inequality regarding W, we infer that:

$$
\begin{aligned}
&\frac{\tau}{2}\int_\Omega |\mu_\tau^k|^2 \,\mathrm{d}x - \frac{\tau}{2}\int_\Omega |\mu_\tau^{k-1}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi_\tau^k|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla\varphi_\tau^{k-1}|^2 \,\mathrm{d}x\\
&\quad + \int_\Omega \Psi(\varphi_\tau^k) \,\mathrm{d}x - \int_\Omega \Psi(\varphi_\tau^{k-1}) \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\sigma_\tau^k|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega |\sigma_\tau^{k-1}|^2 \,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_\Omega |\boldsymbol{v}_\tau^k|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega |\boldsymbol{v}_\tau^{k-1}|^2 \,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla z_\tau^k|^p \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla z_\tau^{k-1}|^p \,\mathrm{d}x\\
&\quad + \int_\Omega \widehat{\beta}_\tau(z_\tau^k) \,\mathrm{d}x - \int_\Omega \widehat{\beta}_\tau(z_\tau^{k-1}) \,\mathrm{d}x + \int_\Omega W(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^k), z_\tau^k) \,\mathrm{d}x\\
&\quad - \int_\Omega W(\varphi_\tau^{k-1}, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) \,\mathrm{d}x + \tau\bigg[\int_\Omega |\nabla\mu_\tau^k|^2 \,\mathrm{d}x + \int_\Omega \frac{|\varphi_\tau^k - \varphi_\tau^{k-1}|^2}{\tau} \,\mathrm{d}x\\
&\quad + \int_\Omega |\nabla\sigma_\tau^k|^2 \,\mathrm{d}x + \int_\Gamma \alpha|\sigma_\tau^k|^2 \,\mathrm{d}\mathcal{H}^{d-1} + C\int_\Omega |\varepsilon(\boldsymbol{v}_\tau^k)|^2 \,\mathrm{d}x + \int_\Omega |D_{\tau,k}z|^2 \,\mathrm{d}x\bigg]\\
&\le \tau\bigg[\int_\Omega U_k\,\mu_\tau^k \,\mathrm{d}x + \int_\Omega S_k\,\sigma_\tau^k \,\mathrm{d}x + \int_\Gamma \alpha\sigma_{\Gamma,\tau}^k\,\sigma_\tau^k \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \pi(z_\tau^{k-1})D_{\tau,k}z \,\mathrm{d}x\bigg]\\
&\le \tau\bigg[\int_\Omega U_k\,\mu_\tau^k \,\mathrm{d}x + C + \int_\Omega |\pi(z_\tau^{k-1})||D_{\tau,k}z| \,\mathrm{d}x\bigg],
\end{aligned}
\tag{4.3.41}
$$

where the latter inequality follows from the fact that S_k, σ_τ^k and $\sigma_{\Gamma,\tau}^k$ are bounded in $L^\infty(\Omega)$ uniformly with respect to k and τ. Then, by the Hölder, Poincaré–Wirtinger, and Young inequalities, recalling that $\|U_k\|_H \le C$, we have

$$
\begin{aligned}
\int_\Omega U_k\,\mu_\tau^k \,\mathrm{d}x &\le \|U_k\|_H\|\mu_\tau^k\|_H \le C\left(\|\mu_\tau^k - \langle\mu_\tau^k\rangle\|_H + |\Omega|^{\frac{1}{2}}|\langle\mu_\tau^k\rangle|\right)\\
&\le C\left(\|\nabla\mu_\tau^k\|_H + |\Omega|^{\frac{1}{2}}|\langle\mu_\tau^k\rangle|\right) \le \eta\int_\Omega |\nabla\mu_\tau^k|^2 \,\mathrm{d}x + C_\eta + C|\langle\mu_\tau^k\rangle|,
\end{aligned}
\tag{4.3.42}
$$

where $\langle\mu_\tau^k\rangle$ denotes the mean value of μ_τ^k and η is a small positive constant yet to be defined. Testing (4.3.2b) with 1 and dividing by $|\Omega|$, we obtain

$$
\langle\mu_\tau^k\rangle = \frac{1}{|\Omega|}\int_\Omega \mu_\tau^k \,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \breve{\Psi}'(\varphi_\tau^k) + \widehat{\Psi}'(\varphi_\tau^{k-1}) + W_{,\varphi}(\varphi_\tau^k, \varepsilon(\boldsymbol{u}_\tau^{k-1}), z_\tau^{k-1}) + \tau D_{\tau,k}\varphi \,\mathrm{d}x.
$$

Adding and subtracting $\widehat{\Psi}'(\varphi_\tau^k)$ and $\mathcal{R}\varphi_\tau^{k-1}$, employing growth assumption (4.1.7) of Ψ, the Lipschitz continuity of $\widehat{\Psi}'$, and the boundedness of h, we have

$$
\begin{aligned}
|\langle\mu_\tau^k\rangle| &\le \frac{1}{|\Omega|}\int_\Omega \left|\breve{\Psi}'(\varphi_\tau^k)+\widehat{\Psi}'(\varphi_\tau^{k-1})\right| + Ch(z_\tau^{k-1})|\varepsilon(\boldsymbol{u}_\tau^{k-1})-\mathcal{R}\varphi_\tau^k| + \tau|D_{\tau,k}\varphi|\,\mathrm{d}x \\
&\le C\Big[\int_\Omega \left|\Psi'(\varphi_\tau^k)\right| + \left|\widehat{\Psi}'(\varphi_\tau^k)-\widehat{\Psi}'(\varphi_\tau^{k-1})\right|\mathrm{d}x \\
&\quad + \int_\Omega h(z_\tau^{k-1})|\varepsilon(\boldsymbol{u}_\tau^{k-1})-\mathcal{R}\varphi_\tau^{k-1}| + h^*|\mathcal{R}\varphi_\tau^{k-1}-\mathcal{R}\varphi_\tau^k| + \tau|D_{\tau,k}\varphi|\,\mathrm{d}x\Big] \\
&\le C\int_\Omega \Psi(\varphi_\tau^k) + h(z_\tau^{k-1})|\varepsilon(\boldsymbol{u}_\tau^{k-1})-\mathcal{R}\varphi_\tau^{k-1}| + \tau|D_{\tau,k}\varphi|\,\mathrm{d}x.
\end{aligned}
\tag{4.3.43}
$$

Using the Young inequality twice and the strong ellipticity of $\mathcal{C}$ from hypothesis (A3), from the above inequality, we obtain

$$
\begin{aligned}
|\langle\mu_\tau^k\rangle| &\le C\int_\Omega \Psi(\varphi_\tau^k) + h(z_\tau^{k-1})|\varepsilon(\boldsymbol{u}_\tau^{k-1})-\mathcal{R}\varphi_\tau^{k-1}|^2\,\mathrm{d}x + \tau\int \eta|D_{\tau,k}\varphi|^2 + C_\eta\,\mathrm{d}x \\
&\le C\int_\Omega \Psi(\varphi_\tau^k) + W(\varphi_\tau^{k-1},\varepsilon(\boldsymbol{u}_\tau^{k-1}),z_\tau^{k-1})\,\mathrm{d}x + \eta\int_\Omega \tau^{-1}|\varphi_\tau^k-\varphi_\tau^{k-1}|^2\,\mathrm{d}x + C_\eta,
\end{aligned}
$$

where η is the same small, positive constant introduced before. So, substituting in (4.3.42), we deduce that

$$
\begin{aligned}
\int_\Omega U_k\,\mu_\tau^k\,\mathrm{d}x &\le C\int_\Omega \Psi(\varphi_\tau^k) + W(\varphi_\tau^{k-1},\varepsilon(\boldsymbol{u}_\tau^{k-1}),z_\tau^{k-1})\,\mathrm{d}x \\
&\quad + \eta\int_\Omega |\nabla\mu_\tau^k|^2\,\mathrm{d}x + \eta\int_\Omega \tau^{-1}|\varphi_\tau^k-\varphi_\tau^{k-1}|^2\,\mathrm{d}x + C_\eta.
\end{aligned}
\tag{4.3.44}
$$

Moreover, recalling that, by hypothesis (A6), π is Lipschitz continuous, using the Hölder inequality and the Young inequality with a small constant η yet to be defined, we get

$$
\begin{aligned}
&\int_\Omega |\pi(z_\tau^{k-1})||D_{\tau,k}z|\,\mathrm{d}x \le C\int_\Omega (|z_\tau^{k-1}|+1)|D_{\tau,k}z|\,\mathrm{d}x \\
&\quad\le C(\|z_\tau^{k-1}\|_H+1)\|D_{\tau,k}z\|_H \le \eta\|D_{\tau,k}z\|_H^2 + C_\eta(\|z_\tau^{k-1}\|_H^2+1) \\
&\quad\le \eta\|D_{\tau,k}z\|_H^2 + C_\eta\left(\sum_{i=1}^{k-1}\tau\|D_{\tau,i}z\|_H^2+1\right),
\end{aligned}
\tag{4.3.45}
$$

where we have also used the fact that $z_\tau^{k-1} = z_0 + \sum_{i=1}^{k-1}\tau D_{\tau,i}z$ if $k\ge 2$ and $z_\tau^{k-1}=z_0$ if $k=1$. Finally, using inequalities (4.3.44) and (4.3.45) in (4.3.41), moving to the left-hand side the terms with η (fixing η small enough) and summing from $k=1$ to j, we obtain

$$
\begin{aligned}
&\frac{\tau}{2}\int_\Omega |\mu_\tau^j|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\varphi_\tau^j|^2\,\mathrm{d}x + \int_\Omega \Psi(\varphi_\tau^j)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\sigma_\tau^j|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\boldsymbol{v}_\tau^j|^2\,\mathrm{d}x \\
&\quad + \frac{1}{p}\int_\Omega |\nabla z_\tau^j|^p\,\mathrm{d}x + \int_\Omega \widehat{\beta}_\tau(z_\tau^j)\,\mathrm{d}x + \int_\Omega W(\varphi_\tau^j,\varepsilon(\boldsymbol{u}_\tau^j),z_\tau^j)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{j}\Big[\tau\Big(\frac{1}{2}\int_\Omega |\nabla\mu_\tau^k|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega \tau^{-1}|\varphi_\tau^k-\varphi_\tau^{k-1}|^2\,\mathrm{d}x+\int_\Omega|\nabla\sigma_\tau^k|^2\,\mathrm{d}x \qquad (4.3.46)\\
&+\int_\Gamma \alpha|\sigma_\tau^k|^2\,\mathrm{d}\mathcal{H}^{d-1}+C\int_\Omega|\varepsilon(\boldsymbol{v}_\tau^k)|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega|D_{\tau,k}z|^2\,\mathrm{d}x\Big)\Big]\\
&\le C_0+C\sum_{k=1}^{j}\Big[\tau\Big(\int_\Omega\Psi(\varphi_\tau^k)\,\mathrm{d}x+\int_\Omega W(\varphi_\tau^{k-1},\varepsilon(\boldsymbol{u}_\tau^{k-1}),z_\tau^{k-1})\,\mathrm{d}x\\
&+\sum_{i=1}^{k-1}\tau\int_\Omega|D_{\tau,i}z|^2\,\mathrm{d}x\Big)\Big],
\end{aligned}
$$

where C does not depend on the initial data, while

$$
\begin{aligned}
C_0&=\frac{1}{2}\int_\Omega|\nabla\varphi_0|^2\,\mathrm{d}x+\int_\Omega\Psi(\varphi_0)\,\mathrm{d}x+\frac{1}{2}\int_\Omega|\sigma_0|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega|\boldsymbol{v}_0|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla z_0|^p\,\mathrm{d}x\\
&\quad+\int_\Omega\widehat{\beta}_\tau(z_0)\,\mathrm{d}x+\int_\Omega W(\varphi_0,\varepsilon(\boldsymbol{u}_0),z_0)\,\mathrm{d}x\\
&\le C\Big[\|\varphi_0\|_V^2+\|\sigma_0\|_H^2+\|\boldsymbol{v}_0\|_H^2+\|\boldsymbol{u}_0\|_V^2+\|z_0\|_Z^p+\int_\Omega\Psi(\varphi_0)\,\mathrm{d}x+\int_\Omega\widehat{\beta}(z_0)\,\mathrm{d}x\Big].
\end{aligned}
$$

Here we used the fact that $\widehat{\beta}_\tau(z_0)\le\widehat{\beta}(z_0)$ a.e. that comes directly from the definition of $\widehat{\beta}_\tau$ (see Lemma 2.20) and the following inequality regarding the elastic energy

$$
\begin{aligned}
\int_\Omega W(\varphi_0,\varepsilon(\boldsymbol{u}_0),z_0)\,\mathrm{d}x&=\int_\Omega\frac{h(z_0)}{2}\mathcal{C}(\varepsilon(\boldsymbol{u}_0)-\mathcal{R}\varphi_0):(\varepsilon(\boldsymbol{u}_0)-\mathcal{R}\varphi_0)\,\mathrm{d}x\\
&\le Ch^*\int_\Omega|\varepsilon(\boldsymbol{u}_0)-\mathcal{R}\varphi_0|^2\,\mathrm{d}x\le C(\|\boldsymbol{u}_0\|_V^2+\|\varphi_0\|^2).
\end{aligned}
$$

Applying the discrete Gronwall inequality stated in Lemma 2.4 to (4.3.46) leads to the boundedness of the left-hand side, from which we have

$$
\begin{aligned}
&\|\varphi_\tau^j\|_V+\|\Psi(\varphi_\tau^j)\|_{L^1(\Omega)}+\|\sigma_\tau^j\|_H+\|\boldsymbol{v}_\tau^j\|_H\\
&+\|\nabla z_\tau^j\|_{L^p(\Omega)}+\|\widehat{\beta}(z_\tau^j)\|_H+\|\varepsilon(\boldsymbol{u}_\tau^j)\|_H+\sum_{k=1}^{K_\tau}\Big[\int_{t_\tau^{k-1}}^{t_\tau^k}\Big(\|\nabla\mu_\tau^k\|_H^2 \qquad (4.3.47)\\
&+\|\tau^{-1/2}(\varphi_\tau^k-\varphi_\tau^{k-1})\|_H^2+\|\nabla\sigma_\tau^k\|_H^2+\|\varepsilon(\boldsymbol{v}_\tau^k)\|_H^2+\|D_{\tau,k}z\|_H^2\Big)\,\mathrm{d}s\Big]\le C
\end{aligned}
$$

and, as a consequence, (4.3.22) and (4.3.26).

Consequences of the energy estimate. From the equality

$$
z_\tau^j=z_0+\sum_{k=1}^{j}\int_{t_\tau^{k-1}}^{t_\tau^k}D_{\tau,k}z\,\mathrm{d}s,
$$

and (4.3.47), we have $\|z_\tau^j\|_H \le C$. By the Poincaré–Wirtinger inequality,

$$\|z_\tau^j\|_Z \le C\left(\|\nabla z_\tau^j\|_{L^p(\Omega)} + |\langle z_\tau^j\rangle|\right) \le C. \tag{4.3.48}$$

Here we used $\|\nabla z_\tau^j\|_{L^p(\Omega)} \le C$ by (4.3.47) and we controlled the mean value of z_τ^j with its bounded H norm. We can also gain a mean value estimate for μ_τ^k. Combining the first line from (4.3.43) with (4.3.47), we immediately obtain $|\langle \mu_\tau^k\rangle| \le C$. As a consequence, exploiting the Poincaré–Wirtinger inequality, it follows that

$$\|\mu_\tau^j\|_H \le \|\mu_\tau^j - \langle \mu_\tau^j\rangle\|_H + C|\langle \mu_\tau^j\rangle| \le C\left(\|\nabla \mu_\tau^j\|_H + 1\right)$$

and, thanks to (4.3.47), we get (4.3.25). Notice that here we also employ $\underline{\mu}_\tau = 0$ in $[0,\tau]$ to obtain the estimate for $\underline{\mu}_\tau$. Finally, by comparison in (4.3.19a) and (4.3.19c), we have (4.3.23) and (4.3.27).

Higher order estimate for the displacement. We test equation (4.3.2d) with $-\tau\,\mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]$, obtaining

$$\begin{aligned}
&-\int_\Omega (\boldsymbol{v}_\tau^k - \boldsymbol{v}_\tau^{k-1}) \cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x + \tau\int_\Omega \mathrm{div}\left[h(z_\tau^k)\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\\
&\quad + \tau\int_\Omega \mathrm{div}\left[a(z_\tau^k)\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\\
&= \tau\int_\Omega \mathrm{div}\left[h(z_\tau^k)\mathcal{C}\mathcal{R}\varphi_\tau^k\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x.
\end{aligned}$$

Developing the obvious calculations in the second and third terms on the left-hand side and moving some terms to the right-hand side, we have

$$\begin{aligned}
&-\int_\Omega (\boldsymbol{v}_\tau^k - \boldsymbol{v}_\tau^{k-1}) \cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x + \tau\int_\Omega a(z_\tau^k)\,\mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x\\
&= -\tau\int_\Omega \left(h'(z_\tau^k)\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\nabla z_\tau^k\right)\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x - \tau\int_\Omega h(z_\tau^k)\,\mathrm{div}\left[\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x\\
&\quad - \tau\int_\Omega \left(a'(z_\tau^k)\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\nabla z_\tau^k\right)\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x + \tau\int_\Omega \mathrm{div}\left[h(z_\tau^k)\mathcal{C}\mathcal{R}\varphi_\tau^k\right]\cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\mathrm{d}x.
\end{aligned}$$

Concerning the first term on the left-hand side, recall that for every k it holds $\boldsymbol{u}_\tau^k = 0$ on Γ and, consequently, $\boldsymbol{v}_\tau^k = 0$ on Γ. Thus, it can be estimated as follows:

$$\begin{aligned}
&-\int_\Omega (\boldsymbol{v}_\tau^k - \boldsymbol{v}_\tau^{k-1}) \cdot \mathrm{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x = \int_\Omega \left[\varepsilon(\boldsymbol{v}_\tau^k) - \varepsilon(\boldsymbol{v}_\tau^{k-1})\right] : \left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\\
&\ge \frac{1}{2}\int_\Omega \varepsilon(\boldsymbol{v}_\tau^k) : \mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \varepsilon(\boldsymbol{v}_\tau^{k-1}) : \mathcal{V}\varepsilon(\boldsymbol{v}_\tau^{k-1})\,\mathrm{d}x,
\end{aligned} \tag{4.3.49}$$

where the inequality holds because $\mathcal{V}$ is symmetric and positive-definite, so the associated quadratic form is convex. For the second left-hand term, since a is bounded from below by a strictly positive constant by hypothesis (A4), using Lemma 2.23 (and the fact that $\boldsymbol{v}_\tau^k = 0$ on Γ), we obtain

$$\begin{aligned}\tau \int_\Omega \left(a(z_\tau^k)\operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\right)\cdot \operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x \\ \geq a_*\tau \int_\Omega \left|\operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\right|^2 \mathrm{d}x \geq C_*\tau\|\boldsymbol{v}_\tau^k\|_W^2.\end{aligned} \tag{4.3.50}$$

The first term on the right-hand side can be estimated as follows:

$$\begin{aligned}&\left| -\tau \int_\Omega \left(h'(z_\tau^k)\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\nabla z_\tau^k\right)\cdot \operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\right| \\ &\quad\leq C\tau\|\varepsilon(\boldsymbol{u}_\tau^k)\|_{L^q(\Omega)}\|\nabla z_\tau^k\|_{L^p(\Omega)}\|\operatorname{div}[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)]\|_H,\end{aligned}$$

thanks to Hölder inequality. Here q is chosen to satisfy $\frac{1}{q}+\frac{1}{p}+\frac{1}{2}=1$ and, since $p>d$ and $d=2$ or $d=3$, it is easy to check that $q\in[2,6)$. So, because of the embedding $V \hookrightarrow L^q(\Omega)$, Lemma 2.23, the energy estimate (4.3.47) and the Young inequality, it follows:

$$\left| -\tau \int_\Omega \left(h'(z_\tau^k)\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\nabla z_\tau^k\right)\cdot \operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\right| \leq C_\eta\tau\|\boldsymbol{u}_\tau^k\|_W^2+\eta\tau\|\boldsymbol{v}_\tau^k\|_W^2, \tag{4.3.51}$$

where $\eta>0$ is small and yet to be chosen.
Regarding the second term on the right-hand side, since h is bounded, we deduce that

$$\begin{aligned}&\left| -\tau \int_\Omega \left(h(z_\tau^k)\operatorname{div}\left[\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)\right]\right)\cdot \operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\right| \\ &\quad\leq C\tau\|\operatorname{div}[\mathcal{C}\varepsilon(\boldsymbol{u}_\tau^k)]\|_H\|\operatorname{div}[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)]\|_H \leq C\tau\|\boldsymbol{u}_\tau^k\|_W\|\boldsymbol{v}_\tau^k\|_W \\ &\quad\leq \tau C_\eta\|\boldsymbol{u}_\tau^k\|_W^2+\eta\tau\|\boldsymbol{v}_\tau^k\|_W^2.\end{aligned} \tag{4.3.52}$$

We handle the third term on the right-hand side using the fact that a is Lipschitz continuous by hypothesis (A4), the Hölder inequality, previous estimates, the Young inequality, the embedding $V \hookrightarrow L^q(\Omega)$, and Ehrling's Lemma stated in Theorem 2.7, obtaining

$$\begin{aligned}&\left| -\tau \int_\Omega \left[a'(z_\tau^k)\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\nabla z_\tau^k\right]\cdot \operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\right| \\ &\quad\leq C\tau\|\varepsilon(\boldsymbol{v}_\tau^k)\|_{L^q(\Omega)}\|\nabla z_\tau^k\|_{L^p(\Omega)}\|\operatorname{div}[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)]\|_H \\ &\quad\leq C\tau\|\varepsilon(\boldsymbol{v}_\tau^k)\|_{L^q(\Omega)}\|\boldsymbol{v}_\tau^k\|_W \leq C_\eta\tau\|\varepsilon(\boldsymbol{v}_\tau^k)\|_{L^q(\Omega)}^2+\eta\|\boldsymbol{v}_\tau^k\|_W^2 \\ &\quad\leq \eta\tau\|\boldsymbol{v}_\tau^k\|_W^2+\left(\theta\tau\|\boldsymbol{v}_\tau^k\|_W^2+C_{\eta,\theta}\tau\|\boldsymbol{v}_\tau^k\|_H^2\right)\leq(\eta+\theta)\tau\|\boldsymbol{v}_\tau^k\|_W^2+C_{\eta,\theta}\tau\end{aligned} \tag{4.3.53}$$

where $\eta,\theta>0$ are small and yet to be chosen.
Finally, we turn our attention to the last term on the right-hand side. After noticing that

$$\operatorname{div}\left[h(z_\tau^k)\mathcal{C}\mathcal{R}\varphi_\tau^k\right]=\mathcal{C}\mathcal{R}\left(h'(z_\tau^k)\varphi_\tau^k\nabla z_\tau^k+h(z_\tau^k)\nabla\varphi_\tau^k\right)+h(z_\tau^k)\varphi_\tau^k\operatorname{div}(\mathcal{C}\mathcal{R}),$$

we use the Hölder and the Young inequalities and the previous estimates. We get

$$\begin{aligned}&\left|\tau\int_\Omega \operatorname{div}\left[h(z_\tau^k)\mathcal{C}\mathcal{R}\varphi_\tau^k\right]\cdot\operatorname{div}\left[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\right]\,\mathrm{d}x\right|\\ &\quad\le C\tau\left(\|\varphi_\tau^k\|_{L^q(\Omega)}\|\nabla z_\tau^k\|_{L^p(\Omega)}+\|\nabla\varphi_\tau^k\|_H+\|\varphi_\tau^k\|_H\right)\|\operatorname{div}[\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)]\|_H\\ &\quad\le C\tau\|\boldsymbol{v}_\tau^k\|_W\le\eta\tau\|\boldsymbol{v}_\tau^k\|_W^2+C_\eta\tau.\end{aligned}\tag{4.3.54}$$

Combining (4.3.49)–(4.3.54) and fixing η and θ small enough lead to

$$\frac12\int_\Omega\varepsilon(\boldsymbol{v}_\tau^k):\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^k)\,\mathrm{d}x-\frac12\int_\Omega\varepsilon(\boldsymbol{v}_\tau^{k-1}):\mathcal{V}\varepsilon(\boldsymbol{v}_\tau^{k-1})\,\mathrm{d}x+\frac{C_*}{2}\tau\|\boldsymbol{v}_\tau^k\|_W^2\le C\tau(1+\|\boldsymbol{u}_\tau^k\|_W^2).$$

So, summing for $k=1$ to j and recalling that $\mathcal{V}$ is coercive by hypothesis (A3), we get

$$\begin{aligned}&\|\varepsilon(\boldsymbol{v}_\tau^k)\|_H^2+\sum_{k=1}^j\tau\|\boldsymbol{v}_\tau^k\|_W^2\le C_0+C\sum_{k=1}^j\tau\|\boldsymbol{u}_\tau^k\|_W^2\\ &\quad\le C_0+C\sum_{k=1}^j\tau\left[\sum_{l=1}^k\tau\|\boldsymbol{v}_\tau^l\|_W^2+\|\boldsymbol{u}_0\|_W^2\right]=C_0+C\sum_{k=1}^j\tau\left[\sum_{l=1}^k\tau\|\boldsymbol{v}_\tau^l\|_W^2\right],\end{aligned}$$

where the last equality holds, changing the constant C_0. So, applying the discrete Gronwall inequality stated in Lemma 2.4 leads to

$$\|\varepsilon(\boldsymbol{v}_\tau^k)\|_H^2+\sum_{k=1}^j\tau\|\boldsymbol{v}_\tau^k\|_W^2\le C.$$

Since we already know that $\|\boldsymbol{v}_\tau^k\|_H\le C$ thanks to (4.3.46), (4.3.30) follows. Moreover, recalling the trivial identity

$$\boldsymbol{u}_\tau^k=\boldsymbol{u}_0+\sum_{l=1}^k\tau\boldsymbol{v}_\tau^l,$$

also (4.3.28) holds true. Finally, by equation (4.3.19d), we can write $\partial_t\boldsymbol{v}_\tau$ as

$$\partial_t\boldsymbol{v}_\tau=\operatorname{div}\left[h(\overline{z}_\tau)\mathcal{C}\varepsilon(\overline{\boldsymbol{u}}_\tau)\right]-\operatorname{div}\left[h(\overline{z}_\tau)\mathcal{C}\mathcal{R}\overline{\varphi}_\tau\right]+\operatorname{div}\left[a(\overline{z}_\tau)\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau)\right]$$

and deduce, by comparison, that $\partial_t\boldsymbol{v}_\tau$ is uniformly bounded in $L^2(0,T;H)$. Indeed, h, a, $\mathcal{C}$ and $\mathcal{V}$ are bounded and Lipschitz continuous. The term $\|\nabla\overline{z}_\tau\|_{L^\infty(L^p(\Omega))}$ is uniformly bounded thanks to (4.3.47). Moreover,

$$\|\varepsilon(\overline{\boldsymbol{u}}_\tau)\|_{L^\infty(V)}+\|\overline{\varphi}_\tau\|_{L^\infty(V)}+\|\varepsilon(\overline{\boldsymbol{v}}_\tau)\|_{L^2(V)}\le C$$

thanks to estimates (4.3.28), (4.3.47), (4.3.30), and $V\hookrightarrow L^q(\Omega)$ were q is the Hölder conjugate of $\frac{2p}{p+2}$. So, estimate (4.3.31) follows. Since $\partial_t\boldsymbol{u}_\tau=\overline{\boldsymbol{v}}_\tau$, (4.3.28) and (4.3.30) imply (4.3.29).

Higher order estimate for the order parameter. Equation (4.3.19b) can be rewritten as

$$\breve{\Psi}'(\overline{\varphi}_\tau) - \Delta\overline{\varphi}_\tau = \overline{\mu}_\tau - \widehat{\Psi}'(\underline{\varphi}_\tau) + h(\underline{z}_\tau)\mathcal{C}(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) : \mathcal{R} - (\overline{\varphi}_\tau - \underline{\varphi}_\tau).$$

The right-hand side belongs to $L^2(0,T;H)$ because $\widehat{\Psi}'$ is Lipschitz continuous by hypothesis (A2), h and $\mathcal{C}$ are bounded by hypotheses (A4) and (A3). More specifically, the following estimate holds:

$$\begin{aligned}\|\breve{\Psi}'(\overline{\varphi}_\tau) - \Delta\overline{\varphi}_\tau\|_{L^2(H)} \le \|\overline{\mu}_\tau\|_{L^2(H)} &+ C\left(\|\underline{\varphi}_\tau\|_{L^2(H)} + 1\right)\\ &+ C\left(\|\varepsilon(\underline{\boldsymbol{u}}_\tau)\|_{L^2(H)} + \|\overline{\varphi}_\tau\|_{L^2(H)}\right) + \|\overline{\varphi}_\tau - \underline{\varphi}_\tau\|_{L^2(H)} \le C.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}&\|\breve{\Psi}'(\overline{\varphi}_\tau) - \Delta\overline{\varphi}_\tau\|^2_{L^2(H)}\\ &\quad = \|\breve{\Psi}'(\overline{\varphi}_\tau)\|^2_{L^2(H)} + \|-\Delta\overline{\varphi}_\tau\|^2_{L^2(H)} + 2\int_0^t\!\!\int_\Omega -\Delta\overline{\varphi}_\tau\,\breve{\Psi}'(\overline{\varphi}_\tau)\,\mathrm{d}x\,\mathrm{d}s \qquad (4.3.55)\\ &\quad \ge \|\breve{\Psi}'(\overline{\varphi}_\tau)\|^2_{L^2(H)} + \|-\Delta\overline{\varphi}_\tau\|^2_{L^2(H)},\end{aligned}$$

from which estimate (4.3.24) follows. Observe that the inequality in (4.3.55) holds because $\breve{\Psi}'$ is an increasing continuous function and, therefore, a maximal monotone graph. More explicitly, if we consider its Yosida approximation $\breve{\Psi}'_\delta$, we have

$$\int_0^t\!\!\int_\Omega -\Delta\overline{\varphi}_\tau\,\breve{\Psi}'_\delta(\overline{\varphi}_\tau)\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\!\!\int_\Omega \breve{\Psi}''_\delta(\overline{\varphi}_\tau)|\nabla\overline{\varphi}_\tau|^2\,\mathrm{d}x\,\mathrm{d}s \ge 0,$$

because $\breve{\Psi}'_\delta$ is monotone and Lipschitz continuous, so $\breve{\Psi}''_\delta$ exists a.e. and it is non-negative. Moreover, $\breve{\Psi}'_\delta(\overline{\varphi}_\tau) \to \breve{\Psi}'(\overline{\varphi}_\tau)$ strongly in $L^2(0,T;H)$ as $\delta \to 0^+$ (see Proposition 2.16). So, passing to the limit in the previous expression, we deduce what we claimed. Taking into account (4.3.47), we deduce that (4.3.21) holds. Notice that the asymmetry between $\overline{\varphi}_\tau$ and $\underline{\varphi}_\tau$ in estimate (4.3.21) is a consequence of the fact that $\underline{\varphi}_\tau = \varphi_0$ in $[0,\tau]$ and φ_0 belongs to V, not to W.

More estimates for the damage. From (4.3.19e), we have

$$-\Delta_p\overline{z}_\tau + \beta_\tau(\overline{z}_\tau) = -\partial_t z_\tau - \pi(\underline{z}_\tau) - \frac{\breve{h}'(\overline{z}_\tau) + \widehat{h}'(\underline{z}_\tau)}{2}\mathcal{C}[\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau] : [\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau]$$

in $L^2(0,T;H)$. More specifically, we know that

$$\begin{aligned}&\|-\Delta_p\overline{z}_\tau + \beta_\tau(\overline{z}_\tau)\|_{L^2(H)}\\ &\quad\le \|\partial_t z_\tau\|_{L^2(H)} + C\left(\|\underline{z}_\tau\|_{L^2(H)} + 1 + \|\varepsilon(\underline{\boldsymbol{u}}_\tau)\|^2_{L^4(L^4(\Omega))} + \|\overline{\varphi}_\tau\|^2_{L^4(L^4(\Omega))}\right)\\ &\quad\le C\left(\|\partial_t z_\tau\|_{L^2(H)} + \|\underline{z}_\tau\|_{L^2(H)} + \|\underline{\boldsymbol{u}}_\tau\|^2_{L^\infty(W_0)} + \|\overline{\varphi}_\tau\|^2_{L^\infty(V)} + 1\right) \le C,\end{aligned}$$

making use of the previous estimates, hypothesis (A6) according to which π is Lipschitz continuous, the fact that $\breve{h}$ and $\widehat{h}$ are continuous, the uniform boundedness of $\|\overline{z}_\tau\|_{L^\infty(Q)} + \|\underline{z}_\tau\|_{L^\infty(Q)}$ from (4.3.48), and the embedding $Z \hookrightarrow C^0(\Omega)$. On the other hand,

$$\begin{aligned}
&\|-\Delta_p \overline{z}_\tau + \beta_\tau(\overline{z}_\tau)\|^2_{L^2(H)} \\
&\quad = \|-\Delta_p \overline{z}_\tau\|^2_{L^2(H)} + \|\beta_\tau(\overline{z}_\tau)\|^2_{L^2(H)} + 2\int_0^t\!\!\int_\Omega -\Delta_p \overline{z}_\tau\, \beta_\tau(\overline{z}_\tau)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \|-\Delta_p \overline{z}_\tau\|^2_{L^2(H)} + \|\beta_\tau(\overline{z}_\tau)\|^2_{L^2(H)} + 2\int_0^t\!\!\int_\Omega \beta'_\tau(\overline{z}_\tau)|\nabla \overline{z}_\tau|^p\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \geq \|-\Delta_p \overline{z}_\tau\|^2_{L^2(H)} + \|\beta_\tau(\overline{z}_\tau)\|^2_{L^2(H)},
\end{aligned}$$

where the inequality stands because β'_τ is monotone and Lipschitz continuous (so it is a.e. differentiable with positive derivative). Thus, we have proved (4.3.34) and (4.3.35), employing the fact that $z_0 \in \mathcal{D}(-\Delta_p)$ by hypothesis (A9). Finally, to conclude estimate (4.3.32), we make use of the inequality stated in Lemma 2.21, from which

$$\|\overline{z}_\tau\|_{W^{1+\delta,p}} + \|\underline{z}_\tau\|_{W^{1+\delta,p}} \leq C_\delta \left(\|-\Delta_p \overline{z}_\tau\|_H + \|-\Delta_p \underline{z}_\tau\|_H + \|\overline{z}_\tau\|_H + \|\underline{z}_\tau\|_H\right),$$

for any $\delta \in (0, 1/p)$. Thanks to (4.3.48) that we have already proved, we get (4.3.32). Combining (4.3.32) with the energy estimate (4.3.47), we obtain (4.3.33).

□

4.3.3 Compactness assertions

Lemma 4.17. *There exists a quintuple $(\varphi, \mu, \sigma, \boldsymbol{u}, z)$ that satisfy the regularity of Theorem 4.8 such that, for a non-relabelled subsequence, we have*

$\varphi_\tau \to \varphi$	*weakly-∗*	*in* $L^\infty(0,T;V) \cap H^1(0,T;V')$,	(4.3.56)
	strongly	*in* $C^0([0,T];L^r(\Omega))$,	(4.3.57)
$\overline{\varphi}_\tau \to \varphi$	*weakly-∗*	*in* $L^\infty(0,T;V) \cap L^2(0,T;W)$,	(4.3.58)
$\overline{\varphi}_\tau, \underline{\varphi}_\tau \to \varphi$	*strongly*	*in* $L^r(0,T;L^r(\Omega))$ *and a.e. in* Q,	(4.3.59)
$\breve{\Psi}'(\overline{\varphi}_\tau) \to \breve{\Psi}'(\varphi)$	*weakly*	*in* $L^2(0,T;H)$,	(4.3.60)
$\widehat{\Psi}'(\underline{\varphi}_\tau) \to \widehat{\Psi}'(\varphi)$	*strongly*	*in* $L^2(0,T;H)$,	(4.3.61)
$\overline{\mu}_\tau, \underline{\mu}_\tau \to \mu$	*weakly*	*in* $L^2(0,T;V)$,	(4.3.62)
$\sigma_\tau \to \sigma$	*weakly-∗*	*in* $L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V')$,	(4.3.63)
	strongly	*in* $L^2(0,T;L^r(\Omega))$ *and a.e. in* Q,	(4.3.64)
$\overline{\sigma}_\tau \to \sigma$	*weakly-∗*	*in* $L^\infty(0,T;H) \cap L^2(0,T;V)$,	(4.3.65)
$\boldsymbol{u}_\tau \to \boldsymbol{u}$	*weakly-∗*	*in* $W^{1,\infty}(0,T;V_0) \cap H^1(0,T;W_0)$,	(4.3.66)
	strongly	*in* $C^0([0,T];X)$,	(4.3.67)
$\overline{\boldsymbol{u}}_\tau, \underline{\boldsymbol{u}}_\tau \to \boldsymbol{u}$	*weakly-∗*	*in* $L^\infty(0,T;W_0)$,	(4.3.68)
	strongly	*in* $L^\infty(0,T;X)$,	(4.3.69)
$\boldsymbol{v}_\tau \to \partial_t \boldsymbol{u}$	*weakly-∗*	*in* $L^\infty(0,T;V_0) \cap H^1(0,T;H)$,	(4.3.70)

$$\overline{\boldsymbol{v}}_\tau \to \partial_t \boldsymbol{u} \qquad \textit{weakly-}* \quad \textit{in } L^\infty(0,T;V_0) \cap L^2(0,T;W_0), \tag{4.3.71}$$
$$z_\tau \to z \qquad \textit{weakly-}* \quad \textit{in } L^\infty(0,T;Z) \cap L^2(0,T;W^{1+\delta,p}(\Omega)) \cap H^1(0,T;H), \tag{4.3.72}$$
$$\qquad \textit{strongly} \quad \textit{in } L^s(0,T;Z) \textit{ and a.e. in } Q, \tag{4.3.73}$$
$$\overline{z}_\tau, \underline{z}_\tau \to z \qquad \textit{weakly-}* \quad \textit{in } L^\infty(0,T;Z) \cap L^2(0,T;W^{1+\delta,p}(\Omega)), \tag{4.3.74}$$
$$\qquad \textit{strongly} \quad \textit{in } L^s(0,T;Z) \textit{ and a.e. in } Q, \tag{4.3.75}$$
$$-\Delta_p \overline{z}_\tau \to -\Delta_p z \qquad \textit{weakly} \quad \textit{in } L^2(0,T;H), \tag{4.3.76}$$
$$\beta_\tau(\overline{z}_\tau) \to \xi \qquad \textit{weakly} \quad \textit{in } L^2(0,T;H) \textit{ with } \xi \in \beta(z), \tag{4.3.77}$$

for any $r \in [1,6)$, $s \in [1,+\infty)$, $\delta \in (0,1/p)$, *and* X *such that* $W_0 \hookrightarrow\hookrightarrow X \hookrightarrow H$.

Proof. Most of the convergences are obvious from Proposition 4.16 and standard compactness results (Banach—Alaoglu theorem and Aubin–Lions theorem); this way, we immediately obtain (4.3.58), (4.3.56)–(4.3.57), (4.3.62)–(4.3.68), (4.3.71)–(4.3.74). In the following, we will prove the other ones, focusing on the case $d = 3$, which is the most challenging due to weaker embeddings and interpolation inequalities available. The case $d = 2$ can be treated in a similar but easier way. Notice that it is easy to identify the limit of a piecewise constant interpolant and its retarded function. For example, let's prove that $\underline{\mu}_\tau$ and $\overline{\mu}_\tau$ converge to the same limit. From (4.3.25), we know that $\underline{\mu}_\tau \to \mu$ and $\overline{\mu}_\tau \to \nu$ weakly in $L^2(0,T;V) \hookrightarrow L^2(0,T;H)$. Moreover, we recall that, by definition,

$$\overline{\mu}_\tau(t) = \underline{\mu}_\tau(t+\tau) \qquad \text{for a.e. } t \in (0, T-\tau). \tag{4.3.78}$$

Take a test function $\rho \in C_c^\infty(\Omega \times (0,T))$. Since it has compact support, there exists a $\epsilon > 0$ such that $\operatorname{supp}(\rho) \subseteq \Omega \times (\epsilon, T-\epsilon)$ and we can assume $2\tau < \epsilon$. By a simple change of variables, taking (4.3.78) into account, we have

$$\begin{aligned}\int_0^t\!\!\int_\Omega \overline{\mu}_\tau \rho \,\mathrm{d}x\,\mathrm{d}t &= \int_\epsilon^{T-\epsilon}\int_\Omega \underline{\mu}_\tau(x,t+\tau)\rho(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\epsilon+\tau}^{T+\tau-\epsilon}\int_\Omega \underline{\mu}_\tau(x,s)\rho(x,s-\tau)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t\!\!\int_\Omega \underline{\mu}_\tau(x,s)\rho(x,s-\tau)\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t\!\!\int_\Omega \mu\rho\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Here we used the fact that $\rho(\cdot, \cdot - \tau)$ is still a test function with compact support in $(\epsilon+\tau, T+\tau-\epsilon) \subseteq (0, T-\tau)$ and then we passed to the limit because we have the product of a weakly convergent sequence and a strongly convergent one in $L^2(0,T;H)$. On the other hand, we also know that

$$\int_0^t\!\!\int_\Omega \overline{\mu}_\tau \rho\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega \nu\rho\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, by uniqueness of the limit and the Fundamental Lemma of the Calculus of Variations, we conclude that $\mu = \nu$. In the following, we will discuss the less straightforward limits of the statement. To prove (4.3.59), we initially show that $\overline{\varphi}_\tau, \underline{\varphi}_\tau \to \varphi$ strongly in $L^2(0,T;H)$. Rewriting the piecewise linear interpolant φ_τ as

$$\varphi_\tau(t) = \underline{\varphi}_\tau(t) + \frac{t - t_{k-1}}{\tau}\left[\overline{\varphi}_\tau(t) - \underline{\varphi}_\tau(t)\right]$$

for every $t \in I_\tau^k$, then

$$\|\varphi_\tau - \underline{\varphi}_\tau\|^2_{L^2(H)} = \sum_{k=1}^{K_\tau} \int_{t_\tau^{k-1}}^{t_\tau^k} \int_\Omega \left(\frac{t-t_{k-1}}{\tau}\right)^2 (\overline{\varphi}_\tau(t) - \underline{\varphi}_\tau(t))^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \|\overline{\varphi}_\tau - \underline{\varphi}_\tau\|^2_{L^2(H)} \leq \tau C \to 0,$$

where the last inequality is due to (4.3.22). On the other hand, by (4.3.57), φ_τ goes to φ strongly in $L^2(0,T;H)$, so also $\underline{\varphi}_\tau \to \varphi$ in $L^2(0,T;H)$ and, using again (4.3.22), the same holds for $\overline{\varphi}_\tau$. We can also deduce that, along a non-relabelled subsequence, $\overline{\varphi}_\tau, \underline{\varphi}_\tau \to \varphi$ a.e. in Q. Since $\|\underline{\varphi}_\tau\|_{L^6(L^6(\Omega))}, \|\overline{\varphi}_\tau\|_{L^6(L^6(\Omega))} \leq C$, as ensured by (4.3.21) and by the embedding $L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;L^6(\Omega))$, and given that $\overline{\varphi}_\tau, \underline{\varphi}_\tau \to \varphi$ pointwise a.e., it follows that $\overline{\varphi}_\tau, \underline{\varphi}_\tau \to \varphi$ in $L^r(0,T;L^r(\Omega))$ for every $r \in [1,6)$, so (4.3.59) holds. From (4.3.24), $\|\breve{\Psi}'(\overline{\varphi}_\tau)\|_{L^2(H)} \leq C$; moreover, $\breve{\Psi}$ is continuous and $\overline{\varphi}_\tau \to \varphi$ a.e., so $\breve{\Psi}'(\overline{\varphi}_\tau) \to \breve{\Psi}'(\varphi)$ a.e. in Q. Hence, we have also (4.3.60). By (4.3.59) and the Lipschitz continuity of $\widehat{\Psi}'$, we get (4.3.61). In order to prove (4.3.69), we start by noticing that for every $t \in I_\tau^k$

$$\boldsymbol{u}_\tau(t) = \underline{\boldsymbol{u}}_\tau(t) + \frac{t-t_{k-1}}{\tau}\left[\overline{\boldsymbol{u}}_\tau(t) - \underline{\boldsymbol{u}}_\tau(t)\right] = \underline{\boldsymbol{u}}_\tau(t) + \frac{t-t_{k-1}}{\tau}\int_{I_\tau^k} \overline{\boldsymbol{v}}_\tau \,\mathrm{d}s,$$

from which, using (4.3.30) and $W_0 \hookrightarrow X$, it follows that

$$\|\boldsymbol{u}_\tau - \underline{\boldsymbol{u}}_\tau\|_X \leq \|\overline{\boldsymbol{v}}_\tau\|_{L^2(X)}\tau^{1/2} \leq C\tau^{1/2} \to 0.$$

Since we already know that $\boldsymbol{u}_\tau \to \boldsymbol{u}$ strongly in $L^\infty(0,T;X)$ by (4.3.67), this inequality leads to (4.3.69). Finally, we prove the convergences regarding the damage. From Aubin–Lions compactness result, $L^2(0,T;W^{1+\delta,p}(\Omega)) \cap H^1(0,T;H) \hookrightarrow\hookrightarrow L^2(0,T;Z)$ so, using (4.3.33) and (4.3.32), along a subsequence $z_\tau \to z$ strongly in $L^2(0,T;Z)$. Since z_τ is bounded in $L^\infty(0,T;Z)$, we obtain (4.3.73). As we have already observed before, for every $t \in I_\tau^k$ it holds

$$z_\tau(t) = \overline{z}_\tau(t) - \frac{t_k - t}{\tau}\int_{I_\tau^k} \partial_t z_\tau \,\mathrm{d}s$$

and, as a consequence,

$$\|z_\tau - \overline{z}_\tau\|_{L^\infty(H)} \leq \|\partial_t z_\tau\|_{L^2(H)}\tau^{1/2} \leq C\tau^{1/2} \to 0.$$

Hence, we deduce that, along a subsequence, $z_\tau - \overline{z}_\tau \to 0$ a.e. in Q. Since we know that

$$\|z_\tau - \overline{z}_\tau\|_{L^\infty(Q)} \leq C\|z_\tau\|_{L^\infty(Z)} + \|\overline{z}_\tau\|_{L^\infty(Z)} \leq C,$$

we obtain that $z_\tau - \overline{z}_\tau \to 0$ strongly in $L^s(0,T;L^s)$ for every $s \in [0,+\infty)$. It trivially follows that $z_\tau - \overline{z}_\tau \to 0$ strongly in $L^s(0,T;L^t)$ for every s, $t \in [0,+\infty)$. Now we want to prove that a subsequence converges strongly in $L^2(0,T;Z)$. To reach our purpose, we employ the following inequality of Gagliardo–Nirenberg type for fractional Sobolev spaces (see [BM18, Theorem 1, p. 1356] for further details)

$$\|z_\tau - \overline{z}_\tau\|_Z \le C\|z_\tau - \overline{z}_\tau\|^{\theta}_{L^p(\Omega)}\|z_\tau - \overline{z}_\tau\|^{1-\theta}_{W^{1+\delta,p}}$$

with $\theta = \delta/(1+\delta)$. Taking the square of this inequality, integrating over the time interval $(0,T)$, and using the Hölder inequality leads to

$$\begin{aligned}\int_0^t \|z_\tau - \overline{z}_\tau\|^2_Z \,\mathrm{d}t &\le C\int_0^t \|z_\tau - \overline{z}_\tau\|^{2\theta}_{L^p(\Omega)}\|z_\tau - \overline{z}_\tau\|^{2(1-\theta)}_{W^{1+\delta,p}}\,\mathrm{d}t \\ &\le C\left[\int_0^t \|z_\tau - \overline{z}_\tau\|^{2\theta q}_{L^p(\Omega)}\,\mathrm{d}t\right]^{1/q}\left[\int_0^t \|z_\tau - \overline{z}_\tau\|^{2(1-\theta)q'}_{W^{1+\delta,p}}\,\mathrm{d}t\right]^{1/q'}\end{aligned}$$

where $q = 1/\theta = (1+\delta)/\delta$ and $q' = q/(1-q) = 1/(1-\theta)$ (so that $2\theta q' = 2(1-\theta)q = 2$). Hence, we have

$$\|z_\tau - \overline{z}_\tau\|_{L^2(Z)} \le C\|z_\tau - \overline{z}_\tau\|^{2/q}_{L^2(L^p(\Omega))}\|z_\tau - \overline{z}_\tau\|^{2/q'}_{L^2(W^{1+\delta,p})} \le C\|z_\tau - \overline{z}_\tau\|^{2/q}_{L^2(L^p(\Omega))} \to 0.$$

This strong convergence, combined with the boundedness of $z_\tau - \overline{z}_\tau$ in $L^\infty(0,T;Z)$ (that we have from (4.3.32)), gives us $\|z_\tau - \overline{z}_\tau\|_{L^s(Z)} \to 0$ for every $s \in [0,+\infty)$. Since we already know (4.3.73), we have (4.3.75). Because of (4.3.34), it exists a $w \in L^2(0,T;H)$ such that, along a non-relabeled subsequence, $-\Delta_p\overline{z}_\tau \rightharpoonup w$ in $L^2(0,T;H)$. Then, recalling that $\overline{z}_\tau \to z$ strongly in $L^2(0,T;H)$ and that the operator $-\Delta_p : H \to H$ is maximal monotone so it is strong-weak closed (see Proposition 2.14), we may identify $w = -\Delta_p z$, which proves (4.3.76). Finally, from (4.3.35), we deduce that it exits a $\xi \in L^2(0,T;H)$ such that $\beta_\tau(\overline{z}_\tau) \rightharpoonup \xi$ in $L^2(0,T;H)$. Since β is maximal monotone, β_τ is its Yosida approximation and $\overline{z}_\tau \to z$ strongly in $L^2(0,T;H)$, using [Bar76, Proposition 1.1, p. 42], we deduce that $\xi \in \beta(z)$ so (4.3.77) holds.

□

4.3.4 Passage to the limit in the discrete system

Now we have all the instruments necessary to prove our main result, Theorem 4.8. We want to exploit the compactness result Lemma 4.17, proving that the limit we found is a weak solution to our problem in the sense of Definition 4.5.

Cahn–Hilliard equation. In (4.3.19a), we can easily pass to the weak limit in the terms on the left-hand side and in the second term on the right-hand side using convergence (4.3.56) and (4.3.62). Given a function $\zeta \in L^2(0,T;V)$, we want to prove that

$$\begin{aligned}&\int_0^t\!\!\int_\Omega \left(\frac{\lambda_p\overline{\sigma}_\tau}{1+|W_{,\varepsilon}(\underline{\varphi}_\tau,\varepsilon(\underline{\boldsymbol{u}}_\tau),\underline{z}_\tau)|} - \lambda_a + \overline{f}_\tau\right)g(\underline{\varphi}_\tau,\underline{z}_\tau)\zeta\,\mathrm{d}t\,\mathrm{d}x \\ &\quad\to \int_0^t\!\!\int_\Omega \left(\frac{\lambda_p\sigma}{1+|W_{,\varepsilon}(\varphi,\varepsilon(\boldsymbol{u}),z)|} - \lambda_a + f\right)g(\varphi,z)\zeta\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

First, we note that $\underline{\varphi}_\tau$ (resp. $\underline{z}_\tau$, $\varepsilon(\underline{\boldsymbol{u}}_\tau)$) converges to φ (resp. z, $\varepsilon(\boldsymbol{u})$) a.e. in Q because of (4.3.59) (resp. (4.3.75), (4.3.69)). Since g and $W_{,\varepsilon}$ are continuous, $g(\underline{\varphi}_\tau, \underline{z}_\tau) \to g(\varphi, z)$ and $W_{,\varepsilon}(\underline{\varphi}_\tau, \varepsilon(\underline{\boldsymbol{u}}_\tau), \underline{z}_\tau) \to W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)$ a.e. in Q. Moreover, g is bounded and $1 + |W_{,\varepsilon}| \geq 1$. Finally, we recall that $\overline{\sigma}_\tau \rightharpoonup \sigma$ weakly in $L^2(0,T;H)$ from (4.3.65), and $\overline{f}_\tau \to f$ strongly in $L^2(0,T;H)$, so the above convergence holds. In (4.3.19b), exploiting convergences (4.3.62), (4.3.58), (4.3.60), and (4.3.61), we can immediately pass to the weak limit in all the terms except in $W_{,\varphi}(\overline{\varphi}_\tau, \varepsilon(\underline{\boldsymbol{u}}_\tau), \underline{z}_\tau)$. However, for every $\rho \in L^2(0,T;H)$, we have

$$-\int_0^t\int_\Omega h(\underline{z}_\tau)(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) : \mathcal{C}\mathcal{R}\rho \,\mathrm{d}x\,\mathrm{d}t \to -\int_0^t\int_\Omega h(z)(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : \mathcal{C}\mathcal{R}\rho\,\mathrm{d}x\,\mathrm{d}t,$$

because $\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau \rightharpoonup \varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi$ weakly in $L^2(0,T;H)$ (from (4.3.68) and (4.3.58)) and $\mathcal{C}\mathcal{R}h(\underline{z}_\tau)\rho \to \mathcal{C}\mathcal{R}h(z)\rho$ strongly in $L^2(0,T;H)$. This last convergence holds true since $\mathcal{C}$ is bounded, h is continuous and bounded, $\underline{z}_\tau \to z$ a.e. in Q from (4.3.75), so we can apply the Dominated Convergence Theorem.

Nutrient equation. Rewriting explicitly (4.3.19c), it holds

$$\begin{aligned}\int_0^t \langle \partial_t\sigma_\tau, \zeta\rangle_V \,\mathrm{d}t + \int_0^t\int_\Omega \nabla\overline{\sigma}_\tau\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}t + \alpha\int_0^T\int_\Gamma(\overline{\sigma}_\tau - \overline{\sigma}_{\Gamma,\tau})\zeta\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ = \int_0^t\int_\Omega\left[-\lambda_c\overline{\sigma}_\tau g(\underline{\varphi}_\tau, \underline{z}_\tau) + \Lambda_c(\underline{z}_\tau)(\overline{\sigma}_{c,\tau} - \overline{\sigma}_\tau)\right]\zeta\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for every $\zeta \in L^2(0,T;V)$. As we have already pointed out, $g(\underline{\varphi}_\tau, \underline{z}_\tau)\zeta \to g(\varphi, z)\zeta$ strongly in $L^2(0,T;H)$ and in the same way one can prove that $\Lambda_c(\underline{z}_\tau)\zeta \to \Lambda_c(z)\zeta$ strongly in $L^2(0,T;H)$. So,

$$\begin{aligned}&\int_0^t\int_\Omega\left[-\lambda_c\overline{\sigma}_\tau g(\underline{\varphi}_\tau, \underline{z}_\tau) + \Lambda_c(\underline{z}_\tau)(\overline{\sigma}_{c,\tau} - \overline{\sigma}_\tau)\right]\zeta\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\to \int_0^t\int_\Omega\left[-\lambda_c\sigma g(\varphi, z) + \Lambda_c(z)(\sigma_c - \sigma)\right]\zeta\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

because we also know that $\overline{\sigma}_\tau \rightharpoonup \sigma$ weakly in $L^2(0,T;H)$ thanks to (4.3.65). Regarding the term with the boundary integral, we recall that the trace operator $H^1 \to H_\Gamma$ is linear and continuous. Thus, the weak convergence $\overline{\sigma}_\tau - \overline{\sigma}_{\Gamma,\tau} \rightharpoonup \sigma - \sigma_\Gamma$ in $L^2(0,T;V)$, that we have from (4.3.65) and by construction of $\overline{\sigma}_{\Gamma,\tau}$, leads to the weak convergences of the traces in $L^2(0,T;L^2(\Gamma))$. All the other terms converge using (4.3.63) and (4.3.65). Finally, $0 \leq \sigma \leq M$ because σ_τ satisfies this property and, thanks to (4.3.64), we have pointwise convergence a.e. in Q.

Displacement equation. For every $\boldsymbol{\rho} \in L^2(0,T;H)$, the following equality holds:

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega \partial_t \boldsymbol{v}_\tau \cdot \boldsymbol{\rho} \,\mathrm{d}x\,\mathrm{d}t - \int_0^t\!\!\int_\Omega h'(\overline{z}_\tau)\mathcal{C}\left[\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau\right]\nabla\overline{z}_\tau \cdot \boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^t\!\!\int_\Omega h(\overline{z}_\tau)\,\mathrm{div}\left(\mathcal{C}\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{C}\mathcal{R}\overline{\varphi}_\tau\right)\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\!\!\int_\Omega a'(\overline{z}_\tau)\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau)\nabla\overline{z}_\tau\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^t\!\!\int_\Omega a(\overline{z}_\tau)\,\mathrm{div}\left(\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau)\right)\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

Thanks to (4.3.70), we can pass to the weak limit in the first term. For the other, more complicated addends, we proceed explicitly. Regarding the second term, we want to prove that

$$\int_0^t\!\!\int_\Omega h'(\overline{z}_\tau)\mathcal{C}\left[\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau\right]\nabla\overline{z}_\tau\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega h'(z)\mathcal{C}\left[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi\right]\nabla z\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t.$$

As we have already exploited, $\overline{z}_\tau \to z$ a.e. in Q and h' is continuous, so $h'(\overline{z}_\tau) \to h'(z)$ a.e. in Q. Moreover, from (4.3.32) we know that $\|\overline{z}_\tau\|_{L^\infty(Q)} \le C$ so, since h' is continuous, $\|h'(\overline{z}_\tau)\|_{L^\infty(Q)} \le C$. From (4.3.75), choosing $s = p$, we get that $\nabla\overline{z}_\tau \to \nabla z$ in $L^p(0,T;L^p(\Omega))$. Hence, $h'(\overline{z}_\tau)\nabla\overline{z}_\tau\cdot\boldsymbol{\rho} \to h'(z)\nabla z\cdot\boldsymbol{\rho}$ strongly in $L^q(0,T;L^q(\Omega))$ with $q = \frac{2p}{p+2}$. Let q' be the Hölder conjugate of q, then it is easy to verify that $q' \in [2,6)$ if $d = 3$ and $q' \in [2,+\infty)$ if $d = 2$. From the boundedness of $\mathcal{C}$, (4.3.58) and (4.3.68), we have that $\mathcal{C}\left[\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau\right] \rightharpoonup \mathcal{C}\left[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi\right]$ weakly in $L^{q'}(0,T;L^{q'}(\Omega))$, so the desired convergence follows. To prove that

$$\int_0^t\!\!\int_\Omega h(\overline{z}_\tau)\,\mathrm{div}\left(\mathcal{C}\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{C}\mathcal{R}\overline{\varphi}_\tau\right)\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega h(z)\,\mathrm{div}\left(\mathcal{C}\varepsilon(\boldsymbol{u}) - \mathcal{C}\mathcal{R}\varphi\right)\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t,$$

we observe that h is continuous and bounded and $\overline{z}_\tau \to z$ a.e., so, thanks to the Dominated Convergence Theorem, $h(\overline{z}_\tau)\boldsymbol{\rho} \to h(z)\boldsymbol{\rho}$ in $L^2(0,T;H)$. Moreover, by (4.3.68), (4.3.58), and since $\mathcal{C}$ is bounded and Lipschitz, $\mathrm{div}\left(\mathcal{C}\varepsilon(\overline{\boldsymbol{u}}_\tau) - \mathcal{C}\mathcal{R}\overline{\varphi}_\tau\right) \rightharpoonup \mathrm{div}\left(\mathcal{C}\varepsilon(\boldsymbol{u}) - \mathcal{C}\mathcal{R}\varphi\right)$ weakly in $L^2(0,T;H)$. Now we take into consideration the fourth term, and we are going to show that

$$\int_0^t\!\!\int_\Omega a'(\overline{z}_\tau)\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau)\nabla\overline{z}_\tau\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega a'(z)\mathcal{V}\varepsilon(\boldsymbol{v})\nabla z\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t.$$

Since a' is continuous and $\overline{z}_\tau \to z$ a.e. by (4.3.75), $a'(\overline{z}_\tau) \to a'(z)$ a.e. in Q. Exploiting the boundedness of a' (which follows from the fact that a is Lipschitz), by the Dominated Convergence Theorem $a'(\overline{z}_\tau)\boldsymbol{\rho} \to a'(z)\boldsymbol{\rho}$ strongly in $L^2(0,T;H)$. Moreover, by (4.3.71) and (4.3.58), $\varepsilon(\overline{\boldsymbol{v}}_\tau) \overset{*}{\rightharpoonup} \varepsilon(\partial_t\boldsymbol{u})$ weakly-$*$ in $L^\infty(0,T;H) \cap L^2(0,T;V) \hookrightarrow L^{2p/d}(0,T;L^{2p/(p-2)}(\Omega))$, where the embedding holds true because of the Gagliardo–Nirenberg inequality. More precisely, we apply Theorem 2.5 with

$$r \in \left(2, \frac{2d}{d-2}\right), \qquad q = 2, \qquad s = 1, \qquad \alpha = d\left(\frac{1}{r} - \frac{d-2}{2d}\right).$$

Finally, from (4.3.75) with $s = 2p/(p-d)$, we get $\nabla\overline{z}_\tau \to \nabla z$ strongly in the space $L^{2p/(p-d)}(0,T;L^p(\Omega))$. So, we have concluded, because

$$\frac{1}{2} + \frac{1}{2p/d} + \frac{1}{2p/(p-d)} = 1, \qquad \frac{1}{2} + \frac{1}{2p/(p-2)} + \frac{1}{p} = 1.$$

Lastly, we aim to show that

$$\int_0^t\!\!\int_\Omega a(\overline{z}_\tau)\,\mathrm{div}\,(\mathcal{V}\varepsilon(\overline{\boldsymbol{v}}_\tau))\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^t\!\!\int_\Omega a(z)\,\mathrm{div}\,(\mathcal{V}\varepsilon(\boldsymbol{v}))\cdot\boldsymbol{\rho}\,\mathrm{d}x\,\mathrm{d}t.$$

Continuity and boundedness of a, convergences a.e. of $\overline{z}_\tau$ from (4.3.75), and the Dominated Convergence Theorem lead to $a(\overline{z}_\tau)\boldsymbol{\rho} \to a(z)\boldsymbol{\rho}$ strongly in $L^2(0,T;H)$. From (4.3.71) we have that $\overline{\boldsymbol{v}}_\tau \rightharpoonup \partial_t\boldsymbol{u}$ weakly in $L^2(0,T;W_0)$ and from hypothesis (A4) $\mathcal{V}$ is bounded and Lipschitz continuous. Thus, the last term of the displacement equation passes to the limit.

Damage equation. We discuss only the least immediate term. Consider a test function $\rho \in L^2(0,T;H)$. We will prove that

$$\int_0^t\!\!\int_\Omega \frac{\breve{h}'(\overline{z}_\tau) + \widehat{h}'(\underline{z}_\tau)}{2}(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau) : \mathcal{C}(\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau)\rho\,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_0^t\!\!\int_\Omega \frac{h'(z)}{2}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi) : \mathcal{C}(\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi)\rho\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\overline{z}_\tau$, $\underline{z}_\tau \to z$ a.e. in Ω, $\breve{h}'$, $\widehat{h}'$ are continuous, and $\breve{h}' + \widehat{h}' = h'$, we have $\frac{\breve{h}'(\overline{z}_\tau)+\widehat{h}'(\underline{z}_\tau)}{2} \to \frac{h'(z)}{2}$ a.e. in Ω. Moreover, $\|\overline{z}_\tau\|_{L^\infty(Q)}$, $\|\underline{z}_\tau\|_{L^\infty(Q)}$ are uniformly bounded by (4.3.32). It follows that $\|\breve{h}'(\overline{z}_\tau)\|_{L^\infty(Q)}$, $\|\breve{h}'(\underline{z}_\tau)\|_{L^\infty(Q)} \leq C$. Using the Dominated Convergence Theorem, we deduce that $\frac{\breve{h}'(\overline{z}_\tau)+\widehat{h}'(\underline{z}_\tau)}{2}\rho \to \frac{h'(z)}{2}\rho$ strongly in $L^2(0,T;H)$. From (4.3.69), choosing $X = W^{1,4}(\Omega)$, and from (4.3.59), choosing $r = 4$, we get that $\varepsilon(\underline{\boldsymbol{u}}_\tau) - \mathcal{R}\overline{\varphi}_\tau \to \varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi$ strongly in $L^4(0,T;L^4(\Omega))$. Since $\mathcal{C}$ is bounded, we have the desired convergence.

4.4 Continuous dependence for a modified system

Continuous dependence and uniqueness are not addressed in [Cav25] and remain open problems. The main difficulty to overcome is the p-Laplace operator's degeneracy. To clarify the core of the issue, we proceed as is commonly done when proving continuous dependence. We take the differences of the equations in system (1.3.5), written for two solutions $(\varphi_i, \mu_i, \sigma_i, \boldsymbol{u}_i, z_i)_{i=1,2}$, and we focus on the resulting damage equation, choosing $z := z_1 - z_2$ as a test function. Recalling that $-\Delta_p$ and β are monotone operators, we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_H^2 + \int_\Omega \overbrace{\left[|\nabla z_1|^{p-2}\nabla z_1 - |\nabla z_2|^{p-2}\nabla z_2\right]\cdot\nabla z}^{0\leq}\,\mathrm{d}x + \int_\Omega \overbrace{(\xi_1 - \xi_2)z}^{0\leq}\,\mathrm{d}x$$
$$= -\int_\Omega (\pi(z_1) - \pi(z_2))\,z\,\mathrm{d}x - \int_\Omega (W_{,z}(\varphi_1, \varepsilon(\boldsymbol{u}_1), z_1) - W_{,z}(\varphi_2, \varepsilon(\boldsymbol{u}_2), z_2))\,z\,\mathrm{d}x.$$

Thus, on the left-hand side, we lack a ∇z term which, present in a suitable norm, would allow us to absorb similar addends coming from the other equations and, in the end, to apply the Gronwall inequality. To bypass this difficulty, in the literature it is sometimes considered a modified damage equation (1.3.5e), in which the p-Laplacian is replaced by another nondegenerate operator (see [RR14]). Following this approach, in this section, we consider the PDE system

$$\partial_t\varphi - \Delta\mu = U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z), \tag{4.4.1a}$$
$$\mu = -\Delta\varphi + \Psi'(\varphi) + W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z), \tag{4.4.1b}$$
$$\partial_t\sigma - \Delta\sigma = S(\varphi, \sigma, z), \tag{4.4.1c}$$
$$\partial_{tt}\boldsymbol{u} - \operatorname{div}\left[a(z)\mathcal{V}\varepsilon(\partial_t\boldsymbol{u}) + W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)\right] = \boldsymbol{0}, \tag{4.4.1d}$$
$$\partial_t z - \operatorname{div}(\mathbf{d}(\cdot, \nabla z)) + \beta(z) + \pi(z) + W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z) \ni 0, \tag{4.4.1e}$$

coupled with the boundary conditions

$$\partial_{\boldsymbol{\nu}}\varphi = \partial_{\boldsymbol{\nu}}\mu = 0, \tag{4.4.2a}$$
$$\partial_{\boldsymbol{\nu}}\sigma + \alpha(\sigma - \sigma_\Gamma) = 0, \tag{4.4.2b}$$
$$\boldsymbol{u} = \boldsymbol{0}, \tag{4.4.2c}$$
$$\mathbf{d}(\cdot, \nabla z) \cdot \boldsymbol{\nu} = 0, \tag{4.4.2d}$$

and the initial conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}(0) = \boldsymbol{v}_0, \quad z(0) = z_0. \tag{4.4.3}$$

As already pointed out, the only difference with respect to (1.3.5)–(1.3.7) is the presence of the operator $\mathcal{B}z := -\operatorname{div}(\mathbf{d}(\cdot, \nabla z))$ in place of $-\Delta_p z$ in equation (4.4.1e) and, as a consequence, the modified boundary condition (4.4.2d). All the sources, the constants, and assigned functions remain the same, and we assume that hypotheses (A1)–(A4), (A6)–(A9) hold with only one obvious change, which is requiring

$$z_0 \in \mathcal{D}(\mathcal{B}) \tag{4.4.4}$$

instead of $z_0 \in \mathcal{D}(-\Delta_p)$. Hypothesis (A5) is replaced by the following set of assumptions, where we properly introduce $\mathcal{B}$.

(B1) Let $\phi : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be a Carathéodory function, i.e.,

$$x \to \phi(x, \zeta) \text{ is Lebesgue-measurable for all } \zeta \in \mathbb{R}^d, \tag{4.4.5}$$
$$\zeta \to \phi(x, \zeta) \text{ is continuous for a.e. } x \in \Omega. \tag{4.4.6}$$

We assume that for a.e. $x \in \Omega$ the function $\phi(x, \cdot) : \mathbb{R}^d \to [0, +\infty)$

$$\text{belongs to } C^1(\mathbb{R}^d), \tag{4.4.7}$$
$$\text{is convex with } \phi(x, 0) = 0. \tag{4.4.8}$$

We introduce the notation $\mathbf{d} := \nabla_\zeta \phi$ and require that the following growth conditions hold for certain constants C_5, C_6, C_7, and a.e. $x \in \Omega$, for all $\zeta \in \mathbb{R}^d$

$$\phi(x,\zeta) \geq C_5|\zeta|^p - C_6, \tag{4.4.9}$$
$$|\mathbf{d}(x,\zeta)| \leq C_7(1+|\zeta|^{p-1}) \tag{4.4.10}$$

and for an exponent p that satisfies

$$p > d. \tag{4.4.11}$$

We define $\Phi : H \to [0,+\infty]$ as follows

$$\Phi(z) := \begin{cases} \displaystyle\int_\Omega \phi(x, \nabla z(x))\,\mathrm{d}x & \phi(\cdot,\nabla z) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, thanks to hypothesis (B1), we claim that

$$\Phi \text{ is proper, lower semi-continuous, and convex,} \tag{4.4.12}$$
$$\text{with domain } \mathcal{D}(\Phi) = Z. \tag{4.4.13}$$

As a consequence, its subdifferential $\mathcal{B} := \partial\Phi$ is a (single-valued) maximal monotone operator with domain

$$\begin{aligned} \mathcal{D}(\mathcal{B}) &= \{z \in Z : -\operatorname{div}(\mathbf{d}(\cdot,\nabla z)) \in H, \quad \mathbf{d}(\cdot,\nabla z)\cdot\boldsymbol{\nu} = 0\} \\ &= \left\{ z \in Z : \sup_{\omega \in Z\setminus\{0\}} \frac{\int_\Omega \mathbf{d}(x,\nabla z(x))\cdot\nabla\omega(x)\,\mathrm{d}x}{\|\omega\|_H} < +\infty \right\}. \end{aligned} \tag{4.4.14}$$

For every $z \in \mathcal{D}(\mathcal{B})$ it acts as follows

$$\int_\Omega \mathcal{B}z\,\omega\,\mathrm{d}x = \int_\Omega \mathbf{d}(x,\nabla z(x))\cdot\nabla\omega(x)\,\mathrm{d}x \qquad \forall \omega \in Z.$$

Thus, we have that

$$\mathcal{B}z = -\operatorname{div}(\mathbf{d}(\cdot,\nabla z))$$

in the sense of distributions.

Remark 4.18. The prototypical example of $\mathcal{B}$ is the classical p-Laplace operator with homogeneous Neumann boundary conditions. To obtain it, we set

$$\phi_p(x,\zeta) = \frac{1}{p}|\zeta|^p$$

and, consequently, $\mathbf{d}_p(x,\zeta) = |\zeta|^{p-2}\zeta$. This way we have

$$-\Delta_p z := \mathcal{B}z = -\operatorname{div}\left(|\nabla z|^{p-2}\nabla z\right).$$

Another example is the so-called non-degenerate p-Laplace operator, again with homogeneous Neumann boundary conditions. It is defined starting from

$$\phi_p^{nd}(x,\zeta) = \frac{1}{p}(1+|\zeta|^2)^{\frac{p}{2}}$$

and $\mathbf{d}_p^{nd}(x,\zeta) = (1+|\zeta|^2)^{\frac{p-2}{2}}\zeta$. We obtain

$$-\Delta_p^{nd} z \coloneqq \mathcal{B}z = -\operatorname{div}\left((1+|\nabla z|^2)^{\frac{p-2}{2}}\nabla z\right).$$

It is easy to check that both operators satisfy hypothesis (B1).

(B2) We suppose that ϕ is Lipschitz continuous with respect to x and more precisely that there exists a non-negative constant L such that

$$|\phi(x,\zeta) - \phi(y,\zeta)| \le L|x-y|(1+|\zeta|^p) \tag{4.4.15}$$

for all $x, y \in \Omega$ and for all $\zeta \in \mathbb{R}^d$. We require that ϕ satisfies the p-coercivity condition

$$(\mathbf{d}(x,\zeta) - \mathbf{d}(x,\eta)) \cdot (\zeta - \eta) \ge C_6|\zeta-\eta|^p \tag{4.4.16}$$

for a positive constant C_6 and for every $x \in \Omega$, $\zeta, \eta \in \mathbb{R}^d$.

Remark 4.19. Once again, this hypothesis is fulfilled by the p-Laplace operator and by the non-degenerate p-Laplace operator. Even though it will not be necessary for the existence result, it guarantees more regularity for the solution. In fact, if the operator $\mathcal{B}$ satisfies hypotheses (B1)–(B2), then $\mathcal{D}(\mathcal{B}) \subseteq W^{1+\delta,p}(\Omega)$ for all $0 < \delta < \frac{1}{p}$. Moreover, there exists a $C_\delta > 0$ such that, for all $v \in W^{1+\delta,p}(\Omega)$,

$$\|v\|_{W^{1+\delta,p}} \le C_\delta(\|-\Delta_p v\|_H + \|v\|_H). \tag{4.4.17}$$

See [Sav98, Theorem 2, Remark 3.5]) for further details.

Assuming all the hypotheses enlisted above, the following existence result holds.

Theorem 4.20 (Existence)**.** *Let hypotheses (A1)–(A4), (A6)–(A9) and (B1) be satisfied. Then, there exists a weak solution to the PDE system (4.4.1)–(4.4.3), i.e., a quintuple* $(\varphi, \mu, \sigma, \boldsymbol{u}, z)$ *with the following regularity*

$$\varphi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V'), \quad \mu \in L^2(0,T;V),$$
$$\sigma \in L^2(0,T;V) \cap H^1(0,T;V'),$$
$$\boldsymbol{u} \in H^1(0,T;W_0) \cap W^{1,\infty}(0,T;V_0) \cap H^2(0,T;H),$$
$$z \in L^\infty(0,T;Z) \cap H^1(0,T;H)$$

which complies with the initial conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}(0) = \boldsymbol{v}_0, \quad z(0) = z_0 \quad \text{a.e. in } \Omega,$$

and satisfies the following equations a.e. in $(0,T)$*, and for all* $\zeta \in V$*,* $\boldsymbol{\omega} \in V_0$ *and* $\rho \in Z$

$$\langle \partial_t \varphi, \zeta \rangle_V + \int_\Omega \nabla \mu \cdot \nabla \zeta \, \mathrm{d}x = \int_\Omega U(\varphi, \sigma, \varepsilon(\boldsymbol{u}), z) \, \zeta \, \mathrm{d}x, \tag{4.4.18a}$$

$$\int_\Omega \mu \zeta \, \mathrm{d}x = \int_\Omega \nabla \varphi \cdot \nabla \zeta \, \mathrm{d}x + \int_\Omega \Psi'(\varphi) \zeta \, \mathrm{d}x + \int_\Omega W_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}), z) \zeta \, \mathrm{d}x, \tag{4.4.18b}$$

$$\langle \partial_t \sigma, \zeta \rangle_V + \int_\Omega \nabla \sigma \cdot \nabla \zeta \, \mathrm{d}x + \alpha \int_\Gamma (\sigma - \sigma_\Gamma) \zeta \, \mathrm{d}\mathcal{H}^{d-1} = \int_\Omega S(\varphi, \sigma, z) \zeta \, \mathrm{d}x, \tag{4.4.18c}$$

$$\int_\Omega \partial_{tt} \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}x + \int_\Omega [a(z) \mathcal{V} \varepsilon(\partial_t \boldsymbol{u}) + W_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u}), z)] : \varepsilon(\boldsymbol{\omega}) \, \mathrm{d}x = 0, \tag{4.4.18d}$$

$$\begin{aligned} &\int_\Omega \partial_t z \rho \, \mathrm{d}x + \int_\Omega \mathbf{d}(\cdot, \nabla z) \cdot \nabla \rho \, \mathrm{d}x \\ &\quad + \int_\Omega \xi \rho \, \mathrm{d}x + \int_\Omega \pi(z) \rho \, \mathrm{d}x + \int_\Omega W_{,z}(\varphi, \varepsilon(\boldsymbol{u}), z) \rho \, \mathrm{d}x = 0, \end{aligned} \tag{4.4.18e}$$

where

$$\xi \in L^2(0,T;H) \text{ with } \xi \in \beta(z) \text{ a.e. in } Q.$$

Moreover,

$$0 \le \sigma \le M \text{ a.e. in } Q.$$

Finally, if hypothesis (B2) holds, then

$$z \in L^2(0,T;W^{1+\delta,p}) \tag{4.4.19}$$

for all $0 < \delta < 1/p$*.*

The proof is omitted as it is analogous to the one in [Cav25], besides the obvious modifications. The only difference is the use of the more general operator $\mathcal{B}$ instead of $-\Delta_p$. In particular, the p-Laplace operator enjoys the additional hypothesis (B2), thus (1.3.5d) can be rewritten as

$$\begin{aligned} &\partial_{tt} \boldsymbol{u} - a'(z) \mathcal{V} \varepsilon(\partial_t \boldsymbol{u}) \nabla z - a(z) \operatorname{div} [\mathcal{V} \varepsilon(\partial_t \boldsymbol{u})] \\ &\quad - h'(z) \mathcal{C} \left(\varepsilon(\partial_t \boldsymbol{u}) - \mathcal{R} \varphi \right) \nabla z - h(z) \operatorname{div} [\mathcal{C} \varepsilon(\partial_t \boldsymbol{u}) - \mathcal{C} \mathcal{R} \varphi] = \mathbf{0} \end{aligned}$$

and is satisfied pointwise a.e. in Q. This is no longer true in the more general case we are considering, where z does not belong to $W^{1+\delta,p}(\Omega)$ and equation (4.2.1d) is satisfied in a weaker sense.

To derive the continuous dependence property we aim for, it is necessary to introduce more restrictive assumptions.

(C1) We require that $\phi(x,\cdot) \in C^2(\mathbb{R}^d \setminus \{0\})$ for every $x \in \Omega$ and that it complies with the convexity requirement

$$D^2_\zeta \phi(x,\zeta) \eta \cdot \eta \ge C_8 (1 + |\zeta|)^{p-2} |\eta|^2 \tag{4.4.20}$$

for a positive constant C_8, for all $x \in \Omega$, $\zeta \in \mathbb{R}^d \setminus \{0\}$ and $\eta \in \mathbb{R}^d$.

Remark 4.21. As done in [Kne05, Lemma A.1, p. 149], it can be easily proved that hypothesis (C1) implies the existence of a positive constant C_9 such that

$$(\mathbf{d}(x,\zeta)-\mathbf{d}(x,\eta))\cdot(\zeta-\eta)\geq C_9(1+|\zeta|+|\eta|)^{p-2}|\zeta-\eta|^2 \tag{4.4.21}$$

for every $x\in\Omega$ and for every $\zeta,\eta\in\mathbb{R}^d$.

Remark 4.22. Notice that hypothesis (C1) is satisfied by the non-degenerative p-Laplacian but not by the p-Laplacian. In fact, the latter operator satisfies the weaker inequality

$$D^2_\zeta\phi_p(x,\zeta)\eta\cdot\eta\geq C_8|\zeta|^{p-2}|\eta|^2$$

and consequently

$$(\mathbf{d}_p(x,\zeta)-\mathbf{d}_p(x,\eta))\cdot(\zeta-\eta)\geq C_9(|\zeta|+|\eta|)^{p-2}|\zeta-\eta|^2$$

which will not be enough to prove uniqueness (see also Remark 4.24).

(C2) We assume that

$$g \text{ is Lipschitz continuous,} \tag{4.4.22}$$

$$\Lambda_c \text{ is Lipschitz continuous,} \tag{4.4.23}$$

(C3) We ask that the convex part of the potential Ψ satisfies the following growth condition

$$|\breve{\Psi}'(s)-\breve{\Psi}'(r)|\leq C(1+|s|^4+|r|^4)|s-r| \tag{4.4.24}$$

for a certain constant C and for all $s,r\in\mathbb{R}$.

Notice that, since the derivative of the concave part of Ψ is Lipschitz continuous by hypothesis (A2), the very same inequality with a different constant holds also for Ψ'.

(C4) We require that the elastic coefficient satisfies

$$h' \text{ is Lipschitz continuous,} \tag{4.4.25}$$

and that viscous coefficient

$$a \text{ is constant.} \tag{4.4.26}$$

In the following, when we use this hypothesis, we will incorporate the constant a into the viscosity tensor $\mathcal{V}$, to simplify the notation.

Theorem 4.23 (Continuous dependence). *Let hypotheses (A1)–(A4), (A6)–(A9), (B1), and (C1)–(C4) be satisfied. Then, for every pair $\{(\varphi_i,\mu_i,\sigma_i,\boldsymbol{u}_i,z_i)\}_{i=1,2}$ of weak solutions to (4.4.1)–(4.4.3) corresponding to the initial data $\{(\varphi_{0,i},\sigma_{0,i},\boldsymbol{u}_{0,i},\boldsymbol{v}_{0,i},z_{0,i})\}_{i=1,2}$ and to* *the assigned functions* $\{(f_i, \sigma_{c,i}, \sigma_{\Gamma,i})\}_{i=1,2}$, *the following continuous dependence inequality holds*

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{L^\infty(V')\cap L^2(V)} + \|\mu_1 - \mu_2\|_{L^2(V')} + \|\sigma_1 - \sigma_2\|_{L^\infty(H)\cap L^2(V)} \\
&\qquad + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,\infty}(H)\cap H^1(V)} + \|z_1 - z_2\|_{L^\infty(H)\cap L^2(V)} + \|\nabla z_1 - \nabla z_2\|^{p/2}_{L^p(L^p(\Omega))} \\
&\quad \leq C\Big(\|\varphi_{0,1} - \varphi_{0,2}\|_{V'} + \|\sigma_{0,1} - \sigma_{0,2}\|_H + \|\boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}\|_V + \|\boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}\|_H \\
&\qquad + \|z_{0,1} - z_{0,2}\|_H + \|f_1 - f_2\|_{L^2(H)} + \|\sigma_{c,1} - \sigma_{c,2}\|_{L^2(H)} + \|\sigma_{\Gamma,1} - \sigma_{\Gamma,2}\|_{L^2(L^2(\Gamma))}\Big)
\end{aligned}$$

for a positive constant C *that depends on* T*, and on* $\|\varphi_i\|_{L^\infty(L^6(\Omega))}$*,* M_i*,* $\|\varepsilon(\boldsymbol{u}_i)\|_{L^\infty(L^6(\Omega))}$*,* $\|z_i\|_{L^\infty(Q)}$*,* $\|f_i\|_{L^\infty(H)}$ *for* $i = 1, 2$*. In particular, the solution of* (4.4.1) *coupled with the boundary conditions* (4.4.2) *and the initial conditions* (4.4.3) *is unique.*

Remark 4.24. As it will be pointed out throughout the proof, hypothesis (C1) is a crucial assumption because it guarantees that inequality (4.4.21) holds. The fact that the p-Laplace operator does not satisfy it is the reason why we are not able to prove uniqueness in this case.

4.4.1 Proof of the continuous dependence theorem

Consider two pairs $\{(\varphi_i, \mu_i, \sigma_i, \boldsymbol{u}_i, z_i)\}_{i=1,2}$ of weak solution corresponding to the assigned functions $\{(f_i, \sigma_{c,i}, \sigma_{\Gamma,i})\}_{i=1,2}$ and to the initial data $\{(\varphi_{0,i}, \sigma_{0,i}, \boldsymbol{u}_{0,i}, \boldsymbol{v}_{0,i}, z_{0,i})\}_{i=1,2}$. For the sake of brevity, in the following, we will use the shorter notation

$$\begin{aligned}
&\varphi := \varphi_1 - \varphi_2, && \sigma := \sigma_1 - \sigma_2, && \boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2, && z := z_1 - z_2, \\
&\varphi_0 := \varphi_{0,1} - \varphi_{0,2}, && \sigma_0 := \sigma_{0,1} - \sigma_{0,2}, && \boldsymbol{u}_0 := \boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}, && \boldsymbol{v}_0 := \boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2}, \\
&z_0 := z_{0,1} - z_{0,2}, && f := f_1 - f_2, && \sigma_c := \sigma_{c,1} - \sigma_{c,2}, && \sigma_\Gamma := \sigma_{\Gamma,1} - \sigma_{\Gamma,2}.
\end{aligned}$$

Moreover, we introduce

$$\begin{aligned}
&W_{,\varphi,i} := W_{,\varphi}(\varphi_i, \varepsilon(\boldsymbol{u}_i), z_i), && W_{,\epsilon,i} := W_{,\epsilon}(\varphi_i, \varepsilon(\boldsymbol{u}_i), z_i), && W_{,z,i} := W_{,z}(\varphi_i, \varepsilon(\boldsymbol{u}_i), z_i), \\
&g_i := g(\varphi_i, z_i), && \Psi'_i := \Psi'(\varphi_i), && \Lambda_{c,i} := \Lambda_c(z_i), \\
&h_i := h(z_i), && \mathcal{B}_i := \mathcal{B}(z_i), && \pi_i := \pi(z_i),
\end{aligned}$$

and the related differences

$$\begin{aligned}
&\overline{W}_{,\varphi} := W_{,\varphi,1} - W_{,\varphi,2}, && \overline{W}_{,\epsilon} := W_{,\epsilon,1} - W_{,\epsilon,2}, && \overline{W}_{,z} := W_{,z,1} - W_{,z,2}, \\
&\overline{g} := g_1 - g_2, && \overline{\Psi}' := \Psi'_1 - \Psi'_2, && \overline{\Lambda}_c := \Lambda_{c,1} - \Lambda_{c,2}, \\
&\overline{h} := h_1 - h_2, && \overline{\mathcal{B}} := \mathcal{B}_1 - \mathcal{B}_2, && \overline{\pi} := \pi_1 - \pi_2.
\end{aligned}$$

The derivation of the following estimates takes inspiration from [GLS21a, Theorem 4] for the Cahn–Hilliard and nutrient equations, and from [RR14, Theorem 3] for the displacement and damage equations, suitably adapted to our setting.

Cahn–Hilliard equation. In the following, it will be useful to write the equation in a more abstract setting. For this reason, we introduce the Riesz isomorphism $\mathcal{A} : V \to V'$ classically defined as follows:

$$\langle \mathcal{A}u, v \rangle_V := (u, v)_V = \int_\Omega \nabla u \cdot \nabla v + u\, v \,\mathrm{d}x$$

for all $u, v \in V$. We recall that the restriction of $\mathcal{A}$ to W is itself an isomorphism from W onto H given by

$$\mathcal{A}w = -\Delta w + w$$

for all $w \in W$. We report here, for the reader's convenience, some of the properties enjoyed by the operator $\mathcal{A}$:

(P1) $\langle \mathcal{A}u, \mathcal{A}^{-1}v^* \rangle_V = \langle v^*, u \rangle_V$ for all $u \in V$, $v^* \in V'$,

(P2) $\|\mathcal{A}^{-1}v^*\|_V \le \|v^*\|_{V'}$ for all $v^* \in V'$,

(P3) $\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{dt}}\|v^*(t)\|^2_{V'} = \langle \partial_t v^*(t), \mathcal{A}^{-1}v^*(t) \rangle_V$ for all $v^* \in H^1(0,T;V')$ and for a.e. $t \in (0,T)$.

In particular, property (P2), together with the continuous embedding $V \hookrightarrow L^q(\Omega)$ for an exponent $q \in [2,6]$, yields the straightforward inequality

$$\|\mathcal{A}^{-1}v^*\|_{L^q(\Omega)} \le C\|v^*\|_{V'} \qquad \forall v^* \in V', \tag{4.4.27}$$

which will be frequently used in the following. Our aim is to make $\mathcal{A}$ appear in the Cahn–Hilliard equation in place of the Laplace operator. With this end, we add the term μ to both sides of equation (4.4.1a) and φ to both sides of equation (4.4.1b). Moreover, to simplify the notation, we introduce the shifted potential

$$F(\varphi) := \Psi(\varphi) - \frac{1}{2}\varphi^2,$$

and the related term

$$\overline{F}' := F'(\varphi_1) - F'(\varphi_2) = \overline{\Psi}' - \varphi.$$

Notice that F as well as its convex and concave parts enjoy the same properties (A2), (C3) of Ψ. Writing the equations (4.4.18a)–(4.4.18b) for φ_1, μ_1 and φ_2, μ_2 with the modifications we have discussed and taking their difference, we obtain

$$\begin{aligned}\langle \partial_t\varphi, \zeta \rangle_V + \langle \mathcal{A}\mu, \zeta \rangle_V &= \int_\Omega \mu\zeta \,\mathrm{d}x + \int_\Omega \left(\frac{\lambda_p\sigma_1}{1+|W_{,\varepsilon,1}|} - \lambda_a + f_1 \right)\overline{g}\zeta \,\mathrm{d}x \\ &+ \int_\Omega \left(\frac{\lambda_p\sigma}{1+|W_{,\varepsilon,1}|} + f \right) g_2\zeta \,\mathrm{d}x + \int_\Omega \frac{|W_{,\varepsilon,2}| - |W_{,\varepsilon,1}|}{(1+|W_{,\varepsilon,1}|)(1+|W_{,\varepsilon,2}|)}\lambda_p\sigma_2 g_2\zeta \,\mathrm{d}x,\end{aligned} \tag{4.4.28a}$$

$$\begin{aligned}&- \int_\Omega \mu\zeta \,\mathrm{d}x + \langle \mathcal{A}\varphi, \zeta \rangle_V + \int_\Omega \overline{F}'\zeta \,\mathrm{d}x \\ &= \int_\Omega \frac{\overline{h}}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] : \mathcal{R}\zeta \,\mathrm{d}x + \int_\Omega \frac{h_2}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi] : \mathcal{R}\zeta \,\mathrm{d}x,\end{aligned} \tag{4.4.28b}$$

for any $\zeta \in V$. Notice that we have also rewritten the differences between the mass sources in equation (4.4.28a), as well as $\overline{W}_{,\varphi}$ in (4.4.28b), adding and subtracting some addends. We test equation (4.4.28a) with $\mathcal{A}^{-1}\varphi$ and resort to property (P3) and (P1). We test equation (4.4.28b) with φ and exploit the definition of $\mathcal{A}$. Then we sum what we got and notice that the terms $\langle \mathcal{A}\mu, \mathcal{A}^{-1}\varphi\rangle_V$ and $(\mu, \varphi)_H$ cancel out. We obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\varphi\|_{V'}^2 + \|\varphi\|_V^2 + \int_\Omega \overline{F}'\varphi \,\mathrm{d}x = \int_\Omega \mu\mathcal{A}^{-1}\varphi\,\mathrm{d}x \\
&\quad + \int_\Omega \left(\frac{\lambda_p\sigma_1}{1+|W_{,\varepsilon,1}|} - \lambda_a + f_1\right)\overline{g}\mathcal{A}^{-1}\varphi\,\mathrm{d}x + \int_\Omega \left(\frac{\lambda_p\sigma}{1+|W_{,\varepsilon,1}|} + f\right)g_2\mathcal{A}^{-1}\varphi\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{|W_{,\varepsilon,2}| - |W_{,\varepsilon,1}|}{(1+|W_{,\varepsilon,1}|)(1+|W_{,\varepsilon,2}|)}\lambda_p\sigma_2 g_2\mathcal{A}^{-1}\varphi\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\overline{h}}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] : \mathcal{R}\varphi\,\mathrm{d}x + \int_\Omega \frac{h_2}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi] : \mathcal{R}\varphi\,\mathrm{d}x \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
\tag{4.4.29}
$$

First, we observe that we can estimate from below the $\overline{F}'$ term on the left-hand side by exploiting the convex-concave splitting for Ψ, the monotonicity of $\breve{\Psi}'$, and the fact that $\widehat{\Psi}'$ is Lipschitz continuous. Then, we use the interpolation inequality (2.1.1) and the Young inequality with a small constant η yet to be chosen. We end up with:

$$
\begin{aligned}
\int_\Omega \overline{F}'\varphi\,\mathrm{d}x &= \int_\Omega (\breve{\Psi}'(\varphi_1) - \breve{\Psi}'(\varphi_2))\varphi\,\mathrm{d}x + \int_\Omega (\widehat{\Psi}'(\varphi_1) - \widehat{\Psi}'(\varphi_2) - \varphi)\varphi\,\mathrm{d}x \\
&\geq \int_\Omega (\widehat{\Psi}'(\varphi_1) - \widehat{\Psi}'(\varphi_2) - \varphi)\varphi\,\mathrm{d}x \geq -C\|\varphi\|_H^2 \\
&\geq -C\|\varphi\|_V\|\varphi\|_{V'} \geq -\eta\|\varphi\|_V^2 - C_\eta\|\varphi\|_{V'}^2.
\end{aligned}
\tag{4.4.30}
$$

The next step is estimating from above all the terms on the right-hand side of the equality (4.4.29). First, we rewrite I_1 thanks to equation (4.4.28b) with $\mathcal{A}^{-1}\varphi$ as a test function. We exploit property (P1), the growth condition (C3), the fact that h is Lipschitz continuous and bounded, and the boundedness of $\mathcal{C}$ and $\mathcal{R}$. Then, we use the interpolation inequality (2.1.1) and the Hölder inequality, leading to:

$$
\begin{aligned}
I_1 &= \langle \mathcal{A}\varphi, \mathcal{A}^{-1}\varphi\rangle_V + \int_\Omega \left[\overline{F}' - \frac{\overline{h}}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] : \mathcal{R} - \frac{h_2}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi] : \mathcal{R}\right]\mathcal{A}^{-1}\varphi\,\mathrm{d}x \\
&\leq \|\varphi\|_H^2 + C\int_\Omega \left[\left(1 + |\varphi_1|^4 + |\varphi_2|^4\right)|\varphi| + |z|\left(|\varepsilon(\boldsymbol{u}_1)| + |\varphi_1|\right) + |\varepsilon(\boldsymbol{u})|\right]|\mathcal{A}^{-1}\varphi|\,\mathrm{d}x \\
&\leq \|\varphi\|_V\|\varphi\|_{V'} + C\left(1 + \|\varphi_1\|_{L^6(\Omega)}^4 + \|\varphi_2\|_{L^6(\Omega)}^4\right)\|\varphi\|_{L^6(\Omega)}\|\mathcal{A}^{-1}\varphi\|_{L^6(\Omega)} \\
&\quad + C\|z\|_H\left(\|\varepsilon(\boldsymbol{u}_1)\|_{L^6(\Omega)} + \|\varphi_1\|_{L^6(\Omega)}\right)\|\mathcal{A}^{-1}\varphi\|_{L^3(\Omega)} + C\|\varepsilon(\boldsymbol{u})\|_H\|\mathcal{A}^{-1}\varphi\|_H.
\end{aligned}
$$

Recalling inequality (4.4.27), and that $\varphi_1, \varphi_2, \varepsilon(\boldsymbol{u}_1) \in L^\infty(0,T;V)$, we employ the Young inequality choosing η as the small parameter introduced above. We have:

$$
\begin{aligned}
I_1 &\leq C\left(\|\varphi\|_V + \|z\|_H + \|\varepsilon(\boldsymbol{u})\|_H\right)\|\varphi\|_{V'} \\
&\leq \eta\|\varphi\|_V^2 + C_\eta\|\varphi\|_{V'}^2 + C\left(\|z\|_H^2 + \|\varepsilon(\boldsymbol{u})\|_H^2\right).
\end{aligned}
\tag{4.4.31}
$$

Next, we turn our attention to I_2. Keeping in mind that $\sigma_1 \in L^\infty(Q), f_1 \in L^\infty(0,T;H)$, and that g is Lipschitz continuous, we first apply the Hölder and the Young inequalities, and then the interpolation inequalities (2.3.3), (4.4.27). We infer

$$\begin{aligned} I_2 &\le C \int_\Omega (1+|f_1|)\,(|\varphi|+|z|)\,|\mathcal{A}^{-1}\varphi|\,\mathrm{d}x \\ &\le C\|1+f_1\|_{L^\infty(H)}\left(\|\varphi\|_{L^3(\Omega)}+\|z\|_{L^3(\Omega)}\right)\|\mathcal{A}^{-1}\varphi\|_{L^6(\Omega)} \\ &\le C\left(\|\varphi\|_V+\|z\|_{L^3(\Omega)}\right)\|\varphi\|_{V'} \le \eta\left(\|\varphi\|_V^2+\|\nabla z\|_H^2\right)+C_\eta\left(\|\varphi\|_{V'}^2+\|z\|_H^2\right). \end{aligned} \tag{4.4.32}$$

The term I_3 can be handled by g boundedness, Hölder's and Young's inequalities, yielding to

$$\begin{aligned} I_3 &\le C\left(\|\sigma\|_H+\|f\|_H\right)\|\mathcal{A}^{-1}\varphi\|_H \le C\left(\|\sigma\|_H+\|f\|_H\right)\|\varphi\|_{V'} \\ &\le C\left(\|\sigma\|_H^2+\|f\|_H^2+\|\varphi\|_{V'}^2\right). \end{aligned} \tag{4.4.33}$$

Regarding I_4, we notice that

$$\begin{aligned} ||W_{,\varepsilon,2}|-|W_{,\varepsilon,1}|| \le |W_{,\varepsilon,2}-W_{,\varepsilon,1}| &= |-\overline{h}\mathcal{C}[\varepsilon(\boldsymbol{u}_1)-\mathcal{R}\varphi_1]-h_2\mathcal{C}[\varepsilon(\boldsymbol{u})-\mathcal{R}\varphi]| \\ &\le C\Big(|z|(|\varepsilon(\boldsymbol{u}_1)|+|\varphi_1|)+|\varepsilon(\boldsymbol{u})|+|\varphi|\Big), \end{aligned}$$

where the last inequality follows from the boundedness of $\mathcal{C}$, $\mathcal{R}$ and h. Consequently, since σ_2 and g_2 are bounded, proceeding as before, we have

$$\begin{aligned} I_4 &\le C\|z\|_H\left(\|\varepsilon(\boldsymbol{u}_1)\|_{L^6(\Omega)}+\|\varphi_1\|_{L^6(\Omega)}\right)\|\mathcal{A}^{-1}\varphi\|_{L^3(\Omega)} \\ &\quad + C\left(\|\varepsilon(\boldsymbol{u})\|_H+\|\varphi\|_H\right)\|\mathcal{A}^{-1}\varphi\|_H \\ &\le C\left(\|z\|_H+\|\varepsilon(\boldsymbol{u})\|_H+\|\varphi\|_H\right)\|\varphi\|_{V'} \\ &\le C\left(\|\varphi\|_{V'}^2+\|z\|_H^2+\|\varepsilon(\boldsymbol{u})\|_H^2+\|\varphi\|_H^2\right) \\ &\le C\left(\|\varphi\|_{V'}^2+\|z\|_H^2+\|\varepsilon(\boldsymbol{u})\|_H^2\right)+\eta\|\varphi\|_V^2+C_\eta\|\varphi\|_{V'}^2. \end{aligned} \tag{4.4.34}$$

Regarding I_5, we exploit the fact that h is Lipschitz continuous and the boundedness of the tensors $\mathcal{C}, \mathcal{R}$. As before, we have

$$\begin{aligned} I_5 &\le C\int_\Omega |z|(|\varepsilon(\boldsymbol{u}_1)|+|\varphi_1|)|\varphi|\,\mathrm{d}x \\ &\le C\|z\|_H\left(\|\varepsilon(\boldsymbol{u}_1)\|_{L^6(\Omega)}+\|\varphi_1\|_{L^6(\Omega)}\right)\|\varphi\|_{L^3(\Omega)} \\ &\le C\|z\|_H\|\varphi\|_V \le C_\eta\|z\|_H^2+\eta\|\varphi\|_V^2. \end{aligned} \tag{4.4.35}$$

Similarly, we estimate the last addend I_6 as follows:

$$\begin{aligned} I_6 &\le C\int_\Omega \Big(|\varepsilon(\boldsymbol{u})|+|\varphi|\Big)|\varphi|\,\mathrm{d}x \le C\left(\|\varepsilon(\boldsymbol{u})\|_H^2+\|\varphi\|_H^2\right) \\ &\le C\|\varepsilon(\boldsymbol{u})\|_H^2+\eta\|\varphi\|_V^2+C_\eta\|\varphi\|_{V'}^2. \end{aligned} \tag{4.4.36}$$

Going back to (4.4.29), owing to (4.4.30)–(4.4.36), we infer that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\varphi\|_{V'}^2 + \|\varphi\|_V^2 \le \eta\Big(\|\varphi\|_V^2 + \|\nabla z\|_H^2\Big) + C_\eta\left(\|\varphi\|_{V'}^2 + \|z\|_H^2\right) \\ + C\left(\|\sigma\|_H^2 + \|\varepsilon(\boldsymbol{u})\|_H^2 + \|f\|_H^2\right)\end{aligned} \tag{4.4.37}$$

for a certain renamed constant η, small and yet to be chosen. Recall that C_η depends also on $\|\boldsymbol{u}_1\|_{L^\infty(W^{1,6}(\Omega))}$, $\|f_1\|_{L^\infty(H)}$, and $\|\varphi_i\|_{L^\infty(L^6(\Omega))}$ with $i = 1, 2$.

Nutrient equation. Taking the difference of (4.4.18c) written for σ_1 and σ_2 with a test function $\zeta \in V$, we obtain:

$$\begin{aligned}&\langle \partial_t \sigma, \zeta\rangle_V + \int_\Omega \nabla\sigma\cdot\nabla\zeta\,\mathrm{d}x + \alpha\int_\Gamma(\sigma - \sigma_\Gamma)\zeta\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad= \int_\Omega\left[-\lambda_c\sigma_1\overline{g} - \lambda_c\sigma g_2 + \overline{\Lambda}_c(\sigma_{c,1} - \sigma_1) + \Lambda_{c,2}(\sigma_c - \sigma)\right]\zeta\,\mathrm{d}x.\end{aligned}$$

Choosing σ as the test function, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\sigma\|_H^2 + \|\nabla\sigma\|_H^2 + \alpha\|\sigma\|_{L^2(\Gamma)}^2 &= \int_\Omega\left[-\lambda_c\sigma_1\overline{g} + \overline{\Lambda}_c(\sigma_{c,1} - \sigma_1) + \Lambda_{c,2}\sigma_c\right]\sigma\,\mathrm{d}x \\ &\quad - \int_\Omega(\lambda_c g_2 + \Lambda_{c,2})\sigma^2\,\mathrm{d}x + \int_\Gamma \sigma_\Gamma\sigma\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

We recall that $\sigma_1, g_2, \Lambda_{c,2}$ are bounded, and that, by hypothesis (C2), the functions g and Λ_c are Lipschitz continuous. Employing this information and the Young and the Hölder inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\sigma\|_H^2 + \|\nabla\sigma\|_H^2 + \alpha\|\sigma\|_{L^2(\Gamma)}^2 \\ &\quad\le C\left(\int_\Omega(|\varphi| + |z| + |\sigma_c|)\,|\sigma|\,\mathrm{d}x + \int_\Omega|\sigma|^2\,\mathrm{d}x + \int_\Gamma|\sigma_\Gamma||\sigma|\,\mathrm{d}\mathcal{H}^{d-1}\right) \\ &\quad\le C\left(\|\varphi\|_H^2 + \|z\|_H^2 + \|\sigma_c\|_H^2 + \|\sigma\|_H^2 + \|\sigma_\Gamma\|_{L^2(\Gamma)}^2\right) + \frac{\alpha}{2}\|\sigma\|_{L^2(\Gamma)}^2.\end{aligned}$$

We handle the term $\|\varphi\|_H^2$ with the interpolation inequality (2.1.1) and then we employ the Young inequality. Moreover, we move to the left-hand side the term multiplied by $\alpha/2$ and then we get rid of it, since it is nonnegative. We infer that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|\sigma\|_H^2 + \|\nabla\sigma\|_H^2 \\ &\quad\le \eta\|\varphi\|_V^2 + C_\eta\|\varphi\|_{V'}^2 + C\left(\|z\|_H^2 + \|\sigma_c\|_H^2 + \|\sigma\|_H^2 + \|\sigma_\Gamma\|_{L^2(\Gamma)}^2\right).\end{aligned} \tag{4.4.38}$$

Displacement equation. We take the difference of (4.4.18d) written for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ with a test function $\boldsymbol{\omega} \in V_0$ and we rewrite the term $\overline{W}_{,\epsilon}$. We get:

$$\int_\Omega \partial_{tt}\boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}x + \int_\Omega\Big[\mathcal{V}\varepsilon(\partial_t\boldsymbol{u}) + \big(\overline{h}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] + h_2\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi]\big)\Big] : \varepsilon(\boldsymbol{\omega})\,\mathrm{d}x = 0.$$

Choosing $\boldsymbol{\omega} = \partial_t \boldsymbol{u}$ and recalling that $\mathcal{V}$ is strongly elliptic from hypothesis (A3) leads to

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t \boldsymbol{u}\|_H^2 + C_{\mathcal{V}}\|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 \le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t \boldsymbol{u}\|_H^2 + \int_\Omega \mathcal{V}\varepsilon(\partial_t \boldsymbol{u}) : \varepsilon(\partial_t \boldsymbol{u})\,\mathrm{d}x \\ &\quad = -\int_\Omega \left(\overline{h}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] + h_2\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi]\right) : \varepsilon(\partial_t \boldsymbol{u})\,\mathrm{d}x. \end{aligned} \tag{4.4.39}$$

We aim to bound the right-hand side. By (C4), h is Lipschitz continuous and bounded, and, by (A3), the tensors $\mathcal{V}, \mathcal{C}$ are bounded. The Hölder inequality leads to

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t \boldsymbol{u}\|_H^2 + C_{\mathcal{V}}\|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 \\ &\quad \le C \int_\Omega |z|\left(|\varepsilon(\boldsymbol{u}_1)| + |\varphi_1|\right)|\varepsilon(\partial_t \boldsymbol{u})| + \left(|\varepsilon(\boldsymbol{u})| + |\varphi|\right)|\varepsilon(\partial_t \boldsymbol{u})|\,\mathrm{d}x \\ &\quad \le C\left[\|z\|_{L^3(\Omega)}\left(\|\varepsilon(\boldsymbol{u}_1)\|_{L^6(\Omega)} + \|\varphi_1\|_{L^6(\Omega)}\right) + \|\varepsilon(\boldsymbol{u})\|_H + \|\varphi\|_H\right]\|\varepsilon(\partial_t \boldsymbol{u})\|_H. \end{aligned}$$

Recalling that $\varepsilon(\boldsymbol{u}_1), \varphi_1 \in L^\infty(0,T;L^6(\Omega))$, employing inequalities (2.3.3) and (2.1.1), and the Young inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_t \boldsymbol{u}\|_H^2 + C_{\mathcal{V}}\|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 \\ &\quad \le C\left(\|z\|_{L^3(\Omega)} + \|\varepsilon(\boldsymbol{u})\|_H + \|\varphi\|_H\right)\|\varepsilon(\partial_t \boldsymbol{u})\|_H \\ &\quad \le \eta\|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 + C_\eta\left(\|z\|_{L^3(\Omega)}^2 + \|\varepsilon(\boldsymbol{u})\|_H^2 + \|\varphi\|_H^2\right) \\ &\quad \le \eta\left(\|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 + \|\nabla z\|_H^2 + \|\varphi\|_V^2\right) + C_\eta\left(\|z\|_H^2 + \|\varphi\|_{V'}^2 + \|\varepsilon(\boldsymbol{u})\|_H^2\right), \end{aligned} \tag{4.4.40}$$

where η is the already introduced small constant yet to be fixed.

Damage equation. Writing equation (4.4.18e) for z_1 and z_2 and taking their difference, we obtain

$$\int_\Omega \partial_t z \rho\,\mathrm{d}x + \int_\Omega (\mathbf{d}(\cdot,\nabla z_1) - \mathbf{d}(\cdot,\nabla z_2)) \cdot \nabla\rho\,\mathrm{d}x + \int_\Omega \overline{\xi}\rho\,\mathrm{d}x + \int_\Omega \overline{\pi}\rho\,\mathrm{d}x + \int_\Omega \overline{W}_{,z}\rho\,\mathrm{d}x = 0,$$

for any $\rho \in Z$. Then we choose $\rho = z$. We employ the fact that β is monotone, hence the term with $\overline{\xi}$ is nonnegative. Moreover, from the pivotal inequality (4.4.21), we deduce

$$\begin{aligned} &\int_\Omega (\mathbf{d}(\cdot,\nabla z_1) - \mathbf{d}(\cdot,\nabla z_2)) \cdot \nabla z\,\mathrm{d}x \ge C_9 \int_\Omega (1 + |\nabla z_1| + |\nabla z_2|)^{p-2}|\nabla z|^2\,\mathrm{d}x \\ &\quad \ge \frac{C_9}{2}\int_\Omega |\nabla z|^2\,\mathrm{d}x + \frac{C_9}{2}\int_\Omega (1 + |\nabla z|)^{p-2}|\nabla z|^2\,\mathrm{d}x \ge \frac{C_9}{2}\|\nabla z\|_H^2 + \frac{C_9}{2}\|\nabla z\|_{L^p(\Omega)}^p. \end{aligned}$$

Finally, we rewrite $\overline{W}_{,z}$ by adding and subtracting some addends. Combining these elements, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|z\|_H^2 + \frac{C_9}{2}\|\nabla z\|_H^2 + \frac{C_9}{2}\|\nabla z\|_{L^p(\Omega)}^p \\ &\quad \le -\int_\Omega \overline{\pi}\, z\,\mathrm{d}x - \int_\Omega \frac{\overline{h}'}{2}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] : [\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1]z\,\mathrm{d}x \\ &\quad - \int_\Omega \frac{h_2'}{2}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi] : [\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1 + \varepsilon(\boldsymbol{u}_2) - \mathcal{R}\varphi_2]z\,\mathrm{d}x =: I_7 + I_8 + I_9. \end{aligned} \tag{4.4.41}$$

Since π is Lipschitz continuous by hypothesis (A6), we have

$$|I_7| \le C\|z\|_H^2. \tag{4.4.42}$$

Regarding the term I_8, we exploit the fact that h' is Lipschitz continuous by (C4), and the boundedness of the tensors $\mathcal{V}, \mathcal{C}$ by hypothesis (A3). Applying the Hölder inequality and the Young inequalities leads to

$$\begin{aligned}
|I_8| &\le C \int_\Omega |z| \left(|\varepsilon(\boldsymbol{u}_1)|^2 + |\varphi_1|^2\right) |z| \,\mathrm{d}x \\
&\le C\|z\|_H \left(\|\varepsilon(\boldsymbol{u}_1)\|^2_{L^6(\Omega)} + \|\varphi_1\|^2_{L^6(\Omega)}\right) \|z\|_{L^6(\Omega)} \\
&\le C\|z\|_H\|z\|_V \le \eta\|\nabla z\|_H^2 + C_\eta\|z\|_H^2,
\end{aligned} \tag{4.4.43}$$

where, as always, η is a small positive constant. Concerning the last addend I_9, we recall that $z_2 \in L^\infty(0,T;Z) \hookrightarrow L^\infty(Q)$. As a consequence, since h' is continuous, h_2' is bounded. Proceeding as before, we obtain

$$\begin{aligned}
|I_9| &\le C \int_\Omega (|\varepsilon(\boldsymbol{u})| + |\varphi|)\left(|\varepsilon(\boldsymbol{u}_1)| + |\varphi_1| + |\varepsilon(\boldsymbol{u}_2)| + |\varphi_2|\right) |z| \,\mathrm{d}x \\
&\le C\left(\|\varepsilon(\boldsymbol{u})\|_H + \|\varphi\|_H\right) \\
&\quad \times \left(\|\varepsilon(\boldsymbol{u}_1)\|_{L^6(\Omega)} + \|\varphi_1\|_{L^6(\Omega)} + \|\varepsilon(\boldsymbol{u}_2)\|_{L^6(\Omega)} + \|\varphi_2\|_{L^6(\Omega)}\right)\|z\|_{L^3(\Omega)} \\
&\le C\left(\|\varepsilon(\boldsymbol{u})\|_H^2 + \|\varphi\|_H^2 + \|z\|^2_{L^3(\Omega)}\right) \\
&\le \eta\left(\|\varphi\|_V^2 + \|\nabla z\|_H^2\right) + C_\eta\left(\|\varphi\|^2_{V'} + \|z\|_H^2\right) + C\|\varepsilon(\boldsymbol{u})\|_H^2.
\end{aligned} \tag{4.4.44}$$

Using (4.4.42)–(4.4.44) in (4.4.41), we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\|z\|_H^2 + \|\nabla z\|_H^2 + \|\nabla z\|^p_{L^p(\Omega)} \\
&\quad\le \eta\left(\|\varphi\|_V^2 + \|\nabla z\|_H^2\right) + C_\eta\left(\|\varphi\|^2_{V'} + \|z\|_H^2\right) + C\|\varepsilon(\boldsymbol{u})\|_H^2.
\end{aligned} \tag{4.4.45}$$

Conclusion. We sum inequalities (4.4.37), (4.4.38), (4.4.40), (4.4.45) and move the remaining terms multiplied by η to the left-hand side, choosing η small enough. Then, observing that

$$\|\varepsilon(\boldsymbol{u})\|_H^2 \le C\left(\|\varepsilon(\boldsymbol{u}_0)\|_H^2 + \int_0^t \|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 \,\mathrm{d}s\right),$$

we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\varphi\|^2_{V'} + \|\sigma\|_H^2 + \|\partial_t\boldsymbol{u}\|_H^2 + \|z\|_H^2\right) \\
&\qquad + \|\varphi\|_V^2 + \|\sigma\|_V^2 + \|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 + \|\nabla z\|_H^2 + \|\nabla z\|^p_{L^p(\Omega)} \\
&\quad\le C\biggl(\|\varphi\|^2_{V'} + \|\sigma\|_H^2 + \|z\|_H^2 + \int_0^t \|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 \,\mathrm{d}s \\
&\qquad + \|\sigma_c\|_H^2 + \|\sigma_\Gamma\|^2_{L^2(\Gamma)} + \|\boldsymbol{u}_0\|_V^2 + \|f\|_H^2\biggr),
\end{aligned}$$

where the constant C depends on the quantities $\|f_1\|_{L^\infty(H)}$, $\|z_2\|_{L^\infty(Q)}$, $\|\varphi_i\|_{L^\infty(L^6(\Omega))}$, $\|\varepsilon(\boldsymbol{u}_i)\|_{L^\infty(L^6(\Omega))}$ for $i = 1, 2$. Integrating in time for $t \in (0, \tau)$ we have

$$
\begin{aligned}
&\|\varphi\|_{V'}^2 + \|\sigma\|_H^2 + \|\partial_t \boldsymbol{u}\|_H^2 + \|z\|_H^2 \\
&\quad + \int_0^\tau \Big(\|\varphi\|_V^2 + \|\sigma\|_V^2 + \|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 + \|\nabla z\|_H^2 + \|\nabla z\|_{L^p(\Omega)}^p \Big)\,\mathrm{d}t \\
&\quad \le C\bigg(\|\varphi_0\|_{V'}^2 + \|\sigma_0\|_H^2 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{v}_0\|_H^2 + \|z_0\|_H^2 + \|\sigma_c\|_{L^2(H)}^2 + \|\sigma_\Gamma\|_{L^2(L^2(\Gamma))}^2 \\
&\qquad + \|f\|_{L^2(H)}^2 + \int_0^\tau \Big[\|\varphi\|_{V'}^2 + \|z\|_H^2 + \int_0^t \|\varepsilon(\partial_t \boldsymbol{u})\|_H^2 \,\mathrm{d}s \Big]\,\mathrm{d}t \bigg),
\end{aligned}
$$

where C now depends also on T. By the Gronwall inequality, we get

$$
\begin{aligned}
&\|\varphi\|_{L^\infty(V')\cap L^2(V)} + \|\sigma\|_{L^\infty(H)\cap L^2(V)} + \|\boldsymbol{u}\|_{W^{1,\infty}(H)\cap H^1(V)} \\
&\quad + \|z\|_{L^\infty(H)\cap L^2(V)} + \|\nabla z\|_{L^p(L^p(\Omega))}^{p/2} \\
&\quad \le C\Big(\|\varphi_0\|_{V'} + \|\sigma_0\|_H + \|\boldsymbol{u}_0\|_V + \|\boldsymbol{v}_0\|_H + \|z_0\|_H \\
&\qquad + \|\sigma_c\|_{L^2(H)} + \|\sigma_\Gamma\|_{L^2(L^2(\Gamma))} + \|f\|_{L^2(H)} \Big).
\end{aligned}
\tag{4.4.46}
$$

To conclude the proof, we only need to derive the estimate for μ. To this end, we proceed by comparison in equation (4.4.28b). With calculations similar to the ones we have already performed and exploiting estimates (4.4.46), we obtain the following:

$$
\begin{aligned}
\|\mu\|_{V'} &\le \|\mathcal{A}\varphi\|_{V'} + \|\overline{F}'\|_{V'} \\
&\quad + \frac{1}{2}\|\overline{h}\mathcal{C}[\varepsilon(\boldsymbol{u}_1) - \mathcal{R}\varphi_1] : \mathcal{R}\|_{V'} + \frac{1}{2}\|h_2\mathcal{C}[\varepsilon(\boldsymbol{u}) - \mathcal{R}\varphi] : \mathcal{R}\|_{V'} \\
&\le \|\varphi\|_V + \|\overline{F}'\|_{V'} + C\||z|(|\varepsilon(\boldsymbol{u}_1)| + |\varphi_1|)\|_H + C\|\varepsilon(\boldsymbol{u})\|_H + C\|\varphi\|_H \\
&\le C\big(\|\varphi\|_V + \|\overline{F}'\|_{V'} + \|z\|_V + \|\boldsymbol{u}\|_V\big).
\end{aligned}
$$

It only remains to bound the $\overline{F}'$ term. By the embedding $L^{6/5}(\Omega) \hookrightarrow V'$, the growth hypothesis (C3), and the Hölder inequality with exponents $5/4$ and 5, we get

$$
\begin{aligned}
\|\overline{F}'\|_{V'}^{6/5} &\le \|\overline{F}'\|_{L^{6/5}(\Omega)}^{6/5} = \int_\Omega |\overline{F}'|^{6/5}\,\mathrm{d}x \le C\int_\Omega (1 + |\varphi_1|^{24/5} + |\varphi_2|^{24/5})|\varphi|^{6/5}\,\mathrm{d}x \\
&\le C(1 + \|\varphi_1\|_{L^6(\Omega)}^{24/5} + \|\varphi_2\|_{L^6(\Omega)}^{24/5})\|\varphi\|_{L^6(\Omega)}^{6/5} \le C\|\varphi\|_V^{6/5}.
\end{aligned}
\tag{4.4.47}
$$

Thus, we infer that

$$
\|\mu\|_{L^2(V')} \le C\left(\|\varphi\|_{L^2(V)} + \|z\|_{L^2(V)} + \|\boldsymbol{u}\|_{L^2(V)}\right).
$$

By the already proven estimate (4.4.46), the proof is concluded.

Chapter 5

Optimal control for a brain tumor growth model with damage

In this chapter, we investigate the nonlinear PDE system (1.3.14)–(1.3.16), hereafter referred to as the *state system*. For each pair of fixed controls $\boldsymbol{\chi} := (\chi_1, \chi_2)$, representing a combination of chemotherapeutic and lactate-targeting drugs, the first question concerns the well-posedness of the associated initial–boundary value problem. As we will show in Section 5.1, the answer is affirmative. In particular, existence is established by combining a suitable maximum principle with Moser-type estimates to prove the boundedness of φ and σ, together with a fixed-point argument, a time discretization of the displacement equation (1.3.14c), and a Moreau–Yosida approximation of the convex part of the potential Ψ. The main mathematical challenges arise from the nonlinear coupling of the equations and, in particular, from the dependence of the elasticity and viscosity matrices in (1.3.14c) on φ and z. Strengthening the assumptions on the initial data and assigned functions, it is possible to prove more regularity and thus a continuous dependence result. Once well-posedness has been established, it is then natural to introduce the following cost functional:

$$
\begin{aligned}
\mathcal{J}((\varphi,\sigma,\boldsymbol{u},z),\boldsymbol{\chi}) :=& \frac{\alpha_1}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{\alpha_2}{2}\|\varphi(T)-\varphi_\Omega\|^2_H + \alpha_3\int_\Omega \varphi(T)\,\mathrm{d}x \\
&+ \frac{\alpha_4}{2}\|\sigma-\sigma_Q\|^2_{L^2(Q)} + \frac{\alpha_5}{2}\|\sigma(T)-\sigma_\Omega\|^2_H \\
&+ \frac{\alpha_6}{2}\|\sqrt{\gamma(\varphi)}\varepsilon(\boldsymbol{u})\|^2_{L^2(Q)} + \frac{\alpha_7}{2}\|z-z_Q\|^2_{L^2(Q)} \\
&+ \alpha_8\int_\Omega z(T)\,\mathrm{d}x + \frac{\alpha_9}{2}\|\boldsymbol{\chi}\|^2_{L^2(Q)}.
\end{aligned}
\tag{5.0.1}
$$

The non-negative constants $\alpha_1, \dots, \alpha_9$ are weights that cannot simultaneously vanish, while φ_Q, φ_Ω, σ_Q, σ_Ω, z_Q are target functions. More explicitly, the term φ_Q (resp. σ_Q, z_Q) is a desired evolution for the tumor (resp. the lactate, the damage), while φ_Ω (resp. σ_Ω) is a desired final configuration of the tumor (resp. concentration of the lactate). The third and eighth addends measure the size of the tumor and the magnitude of the damage at the end of the treatment. Since the presence of high mechanical stress, especially in certain areas of the brain, can compromise its functionality, we are interested in keeping it low. The sixth term serves this purpose, and γ may be, for instance, the indicator function of a subdomain of Ω where the stress is intended to remain low. Finally, the last addend is a L^2-regularization for the controls. We are interested in studying the minimizers of the cost functional (5.0.1) subject to the PDE system (1.3.14)–(1.3.16) and constrained to a suitable admissible control set. We define it as $\mathcal{U}_{\text{ad}} = \mathcal{U}^1_{\text{ad}} \times \mathcal{U}^2_{\text{ad}}$ with

$$\mathcal{U}^1_{\text{ad}} := \{\chi_1 \in L^2(0,T;V) \cap L^\infty(Q) \,:\, \|\chi_1\|_{L^2(V)} \le C_{\text{ad}},\ \underline{\chi}_1 \le \chi_1 \le \overline{\chi}_1 \text{ a.e.}\},$$
$$\mathcal{U}^2_{\text{ad}} := \{\chi_2 \in L^\infty(Q) \,:\, \underline{\chi}_2 \le \chi_2 \le \overline{\chi}_2 \text{ a.e.}\},$$

where $C_{\text{ad}} > 0$ is a fixed constant and $\underline{\chi}_1$, $\overline{\chi}_1$, $\underline{\chi}_2$, $\overline{\chi}_2 \in L^\infty(Q)$ are given nonnegative threshold functions such that $\mathcal{U}_{\text{ad}}$ is nonempty. The admissible control set $\mathcal{U}_{\text{ad}}$ is a subset of

$$\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2 := \left[L^2(0,T;V) \cap L^\infty(Q)\right] \times L^\infty(Q)$$

equipped with its natural norm that we denote with $\|\cdot\|_{\mathcal{U}}$.

Remark 5.1. Notice that $\mathcal{U}_{\text{ad}}$ is a nonempty, closed, and convex subset of $\mathcal{U}$. Moreover, there exists a positive constant R such that

$$\mathcal{U}_{\text{ad}} \subseteq \mathcal{U}_R := \{\boldsymbol{\chi} \in \mathcal{U} \,:\, \|\boldsymbol{\chi}\|_{\mathcal{U}} < R\}.$$

Thus, the optimal control problem we are interested in can be stated as

> *Minimize the cost functional $\mathcal{J}((\varphi, \sigma, \boldsymbol{u}, z), \boldsymbol{\chi})$ subject to the control constraint $\boldsymbol{\chi} \in \mathcal{U}_{ad}$ and to the state system* (1.3.14)–(1.3.16)

which is a nonlinear and nonconvex minimisation problem subject to PDE constraints, and will be treated through the direct method of the Calculus of Variations (see Section 5.2). Then, a major objective is to derive first-order necessary optimality conditions, expressed in the form of a variational inequality. This is achieved by proving the differentiability of the control-to-state operator, which, in turn, requires linearizing the state system, and then introducing the so-called *adjoint system*. Even though numerical aspects are beyond the scope of this thesis, recall that the necessary conditions for optimality are the starting point for finding a numerical approximation of the optimal controls through gradient-descent-like methods. Among the huge literature regarding optimal control for tumor growth models (see, e.g., [BAD15], [Col+21], [GL16], [GLR18], [EK19], [Sig20], and the references cited therein) we would like to recall [Che+24], where the authors deal with a brain tumor-specific model. The resulting PDE system couples three parabolic equations, one for the tumor cell density, and the other two for the intracellular lactate concentration and the capillary lactate concentration, respectively. Another perspective related to the present work is the one from [GLS21b], in which the authors consider a phase field model of Cahn–Hilliard type taking into account the presence of a nutrient (such as oxygen or glucose) and mechanical effects.

5.1 Analytic results regarding the state system

In this section, we begin by listing the basic assumptions that will be used throughout the chapter and by introducing the notion of weak solution for the state system (1.3.14)–(1.3.16). Then, we prove that weak solutions do exist. Under strengthened hypotheses, we subsequently derive additional regularity and prove the continuous dependence of solutions on the given data (in particular, on the control). All of these results together yield the well-posedness of the system, which constitutes the essential starting point for the analysis of the associated control problem.

5.1.1 Hypotheses

We now introduce a set of basic structural assumptions that will remain in force throughout the chapter.

(A1) We suppose that

$$p,\, g : \mathbb{R}^2 \to \mathbb{R} \text{ are continuous functions,} \tag{5.1.1}$$
$$0 \le p \le p^*, \quad 0 \le g \le g^*, \tag{5.1.2}$$

where p^*, g^* are positive constants, and that

$$N \text{ is a positive constant.} \tag{5.1.3}$$

(A2) We assume that

$$k_1, k_2, S : \mathbb{R}^2 \to \mathbb{R} \text{ are continuous and} \tag{5.1.4}$$
$$0 \le k_1 \le k_1^*, \quad 0 < k_{2*} \le k_2 \le k_2^*, \quad 0 \le S \le S^*, \tag{5.1.5}$$

where k_1^*, k_{2*}, k_2^*, S^* denote fixed positive constants.

(A3) We require that $\mathcal{A} = (a_{ijkh}), \mathcal{B} = (b_{ijkh}) : \mathbb{R}^2 \to \mathbb{R}^{n\times n\times n\times n}$ are symmetric fourth-order tensors such that

$$\mathcal{A},\, \mathcal{B} \text{ are of class } C^1 \text{ and bounded along with their partial derivatives,} \tag{5.1.6}$$
$$\mathcal{A} \text{ is strictly positive definite,} \tag{5.1.7}$$
$$\mathcal{B} \text{ is positive definite.} \tag{5.1.8}$$

Moreover, we assume

$$\boldsymbol{f} \in L^\infty(H). \tag{5.1.9}$$

(A4) We suppose that there exists a $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ such that

$$\mathcal{D}(\widehat{\beta}) \subseteq [0,1], \quad \widehat{\beta} \text{ is proper, l.s.c. and convex} \tag{5.1.10}$$

and we denote its subdifferential by $\beta := \partial\widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$.

(A5) We consider a function $\widehat{\pi} \in C^1(\mathbb{R})$ and we denote by $\pi := \widehat{\pi}'$ its derivative, requiring that

$$\widehat{\pi} \text{ is concave,} \tag{5.1.11}$$
$$\pi \text{ is Lipschitz continuous.} \tag{5.1.12}$$

(A6) We suppose that

$$\iota \in L^\infty(0,T;H). \tag{5.1.13}$$

(A7) We assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is

$$\text{a Carathéodory function.} \tag{5.1.14}$$

Moreover, we require that

$$\begin{aligned} &F(x,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^{n\times n} \to \mathbb{R} \text{ is Lipschitz continuous , i.e.,} \\ &\exists C_F > 0 \text{ s.t. } |F(x,\varphi_1,\epsilon_1) - F(x,\varphi_2,\epsilon_2)| \le C_F\left(|\varphi_1 - \varphi_2| + |\epsilon_1 - \epsilon_2|\right) \end{aligned} \tag{5.1.15}$$

for a.e. $x \in \Omega$, for all $\epsilon_1, \epsilon_2 \in \mathbb{R}^{n\times n}$, $\varphi_1, \varphi_2 \in \mathbb{R}$, and that

$$\hat{F}(x) := F(x,0,\mathbf{0}) \in H. \tag{5.1.16}$$

Remark 5.2. Notice that from the requirements (5.1.15) and (5.1.16) we trivially deduce that

$$|F(x,\varphi,\epsilon)| \le |F(x,\varphi,\epsilon) - F(x,0,\mathbf{0})| + |F(x,0,\mathbf{0})| \le C_F\left(|\varphi| + |\epsilon|\right) + |\hat{F}(x)| \tag{5.1.17}$$

for a.e. $x \in \Omega$ and for all $\varphi \in \mathbb{R}, \epsilon \in \mathbb{R}^{n\times n}$.

In the following, for the sake of brevity, we will omit the dependence of F on the point x in the notation, using $F(\varphi,\epsilon)$ instead of $F(x,\varphi,\epsilon)$.

(A8) Regarding the boundary conditions, we suppose that

$$\sigma_\Gamma \in L^2(0,T;L^2(\Gamma)), \quad 0 \le \sigma_\Gamma \le M_0, \tag{5.1.18}$$

where M_0 is a fixed positive constant.

(A9) The initial conditions satisfy

$$\varphi_0 \in V, \quad 0 \le \varphi_0 \le N, \tag{5.1.19}$$
$$\sigma_0 \in H, \quad 0 \le \sigma_0 \le M_0, \tag{5.1.20}$$
$$\boldsymbol{u}_0 \in V_0, \tag{5.1.21}$$
$$z_0 \in V, \quad \widehat{\beta}(z_0) \in L^1(\Omega). \tag{5.1.22}$$

To establish a continuous dependence result, additional regularity of the solutions is required. This, in turn, demands stronger assumptions on the given functions and on the initial data.

(B1) $p, g \in C^{0,1}(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$,

(B2) $k_1, k_2, S \in C^{0,1}(\mathbb{R}^2)$,

(B3) $\iota \in H^1(0,T;H)$,

(B4) $\varphi_0 \in W$, $\boldsymbol{u}_0 \in W_0$, $z_0 \in W$ and $\beta^0(z_0) \in H$, where β^0 is the minimal section of β.

5.1.2 Definition of weak solution and main results

Definition 5.3. *We say that the quadruple* $(\varphi, \sigma, \boldsymbol{u}, z)$ *is a weak solution to the PDE system* (1.3.14) *endowed with the boundary and initial conditions* (1.3.15)–(1.3.16) *if*

$$\begin{gathered}
\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad 0 \le \varphi \le N, \\
\sigma \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad 0 \le \sigma \le M, \\
\boldsymbol{u} \in W^{1,\infty}(0,T;V_0), \\
z \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),
\end{gathered}$$

where $M = M(M_0, S^*)$, *with*

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad z(0) = z_0$$

a.e. in Ω *and there exists a subgradient*

$$\xi \in L^2(0,T;H), \quad \xi \in \beta(z) \text{ a.e. in } Q$$

such that

$$\int_\Omega [\partial_t \varphi \zeta + \nabla \varphi \cdot \nabla \zeta] \, \mathrm{d}x = \int_\Omega \left[(p(\sigma, z) - \chi_1) \varphi \left(1 - \frac{\varphi}{N}\right) - \varphi g(\sigma, z) \right] \zeta \, \mathrm{d}x, \tag{5.1.23a}$$

$$\begin{aligned}
&\langle \partial_t \sigma, \zeta \rangle_V + \int_\Omega \left[\nabla \sigma \cdot \nabla \zeta + \frac{k_1(\varphi, z)\sigma\zeta}{k_2(\varphi, z) + \sigma} \right] \mathrm{d}x + \int_\Gamma (\sigma - \sigma_\Gamma) \zeta \, \mathrm{d}\mathcal{H}^{d-1} \\
&\quad = \int_\Omega \chi_2 S(\varphi, z) \zeta \, \mathrm{d}x,
\end{aligned} \tag{5.1.23b}$$

$$\int_\Omega [\mathcal{A}(\varphi, z)\varepsilon(\partial_t \boldsymbol{u}) + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{u})] : \varepsilon(\boldsymbol{v}) \, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x, \tag{5.1.23c}$$

$$\int_\Omega [\partial_t z \zeta + \nabla z \cdot \nabla \zeta + \xi \zeta + \pi(z) \zeta] \, \mathrm{d}x = \int_\Omega [\iota - F(\varphi, \varepsilon(\boldsymbol{u}))] \, \zeta \, \mathrm{d}x, \tag{5.1.23d}$$

a.e. in $(0,T)$, *for every* $\zeta \in V$ *and* $\boldsymbol{v} \in V_0$.

Remark 5.4. Notice that, with the regularity we require, a solution of (1.3.14)–(1.3.16) in the sense of Definition 5.3 satisfies equation (1.3.14a) and inclusion (1.3.14d) a.e. in Q.

Theorem 5.5 (Existence)**.** *Under the set of hypotheses (A1)–(A9), for every $\boldsymbol{\chi} \in \mathcal{U}$, the PDE system (1.3.14) endowed with the boundary and initial conditions (1.3.15)–(1.3.16) admits at least one weak solution in the sense of Definition 5.3.*

Remark 5.6. As mentioned in the Introduction, it is possible to allow the tensors $\mathcal{A}$, $\mathcal{B}$ to depend on the lactate σ. However, this would require σ to have the same regularity as φ and z (see Step IV of the existence proof). To ensure this, we would need the additional assumptions $\sigma_0 \in V$ and $\sigma_\Gamma \in H^1(0,T;L^2(\Gamma)) \cap L^2(0,T;H^{1/2}(\Gamma))$. Nevertheless, we do not point out this additional regularity for σ in the statement of the following regularity theorem, also because it is not needed for the continuous dependence result stated in Theorem 5.8.

Imposing the additional hypotheses (B1)–(B4), an improvement of the solution regularity can be achieved.

Theorem 5.7 (Regularity)**.** *Under the set of hypotheses (A1)–(A9) and (B1)–(B4), for every $\boldsymbol{\chi} \in \mathcal{U}_R$, the solution to the PDE system (1.3.14)–(1.3.16) we found in Theorem 5.5 enjoys the further regularity*

$$\varphi \in H^1(0,T;V) \cap L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega)),$$
$$\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{W}_0),$$
$$z \in H^1(0,T;V) \cap L^\infty(0,T;W),$$

and the subgradient satisfies

$$\xi \in L^\infty(0,T;H).$$

Moreover, the following estimate is satisfied

$$\begin{aligned} &\|\varphi\|_{H^1(V)\cap L^\infty(W)\cap L^2(H^3(\Omega))} + \|\sigma\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} \\ &\quad + \|\boldsymbol{u}\|_{W^{1,\infty}(\boldsymbol{W}_0)} + \|z\|_{H^1(V)\cap L^\infty(W)} \le C_R \end{aligned} \tag{5.1.24}$$

for a positive constant C_R that depends on the initial data, the assigned functions, and, in particular, on R.

The improved regularity obtained in Theorem 5.7 is enough to prove the following continuous dependence result.

Theorem 5.8 (Continuous dependence)**.** *Under the set of hypotheses (A1)–(A9) and (B1)–(B4), for every pair $\{(\varphi_i, \sigma_i, \boldsymbol{u}_i, z_i)\}_{i=1,2}$ of weak solutions to (1.3.14)–(1.3.16) having the regularity prescribed in Theorem 5.7 and corresponding to the initial data*

$\{(\varphi_{0,i}, \sigma_{0,i}, \boldsymbol{u}_{0,i}, z_{0,i})\}_{i=1,2}$, the controls $\{\boldsymbol{\chi}_i\}_{i=1,2} \in [\mathcal{U}_R]^2$, and the assigned functions $\{(\boldsymbol{f}_i, \iota_i, \sigma_{\Gamma,i})\}_{i=1,2}$, the following continuous dependence inequality holds

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{L^\infty(H)\cap L^2(V)} + \|\sigma_1 - \sigma_2\|_{L^\infty(H)\cap L^2(V)} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{H^1(V_0)} \\ &\qquad + \|z_1 - z_2\|_{L^\infty(H)\cap L^2(V)} \\ &\quad \le C_R\Big(\|\varphi_{0,1} - \varphi_{0,2}\|_H + \|\sigma_{0,1} - \sigma_{0,2}\|_H + \|\boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}\|_H + \|z_{0,1} - z_{0,2}\|_H \\ &\qquad + \|\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2\|_{L^2(H)} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{L^2(H)} + \|\iota_1 - \iota_2\|_{L^2(H)} + \|\sigma_{\Gamma,1} - \sigma_{\Gamma,2}\|_{L^2(L^2(\Gamma))}\Big) \end{aligned}$$

for a positive constant C_R that only depends on the initial data, the assigned functions, and on R.

In particular, the solution of (1.3.14) coupled with the boundary conditions (1.3.15) and the initial conditions (1.3.16) is unique.

5.1.3 Existence of weak solutions

The proof of Theorem 5.5 will be performed through a Schauder fixed-point argument. To do it properly, the first step consists in proving the existence of weak solutions for an approximate system.

The approximate system. The approximate problem is obtained by replacing the maximal monotone operator β with its Yosida approximation $\beta_\lambda := (\widehat{\beta}_\lambda)'$, where $\lambda \in (0, \lambda^*)$ is intended to go to 0 in the limit. Moreover, we introduce the truncation function

$$\alpha(\varphi) := \begin{cases} \varphi\left(1 - \dfrac{\varphi}{N}\right) & \text{if } 0 \le \varphi \le N, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1.25}$$

and we use it in (1.3.14a) in order to have a bounded term on the right-hand side instead of a quadratic one. Finally, in equation (1.3.14b) we substitute the denominator $k_2(\varphi, z) + \sigma$ with $k_2(\varphi, z) + |\sigma|$ to be sure that it does not vanish. This way, we get the approximate PDE system:

$$\partial_t\varphi - \Delta\varphi = (p(\sigma, z) - \chi_1)\,\alpha(\varphi) - \varphi g(\sigma, z), \tag{5.1.26a}$$

$$\partial_t\sigma - \Delta\sigma + \frac{k_1(\varphi, z)\sigma}{k_2(\varphi, z) + |\sigma|} = \chi_2 S(\varphi, z), \tag{5.1.26b}$$

$$-\operatorname{div}\left[\mathcal{A}(\varphi, z)\varepsilon(\partial_t\boldsymbol{u}) + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{u})\right] = \boldsymbol{f}, \tag{5.1.26c}$$

$$\partial_t z - \Delta z + \beta_\lambda(z) + \pi(z) = \iota - F(\varphi, \varepsilon(\boldsymbol{u})). \tag{5.1.26d}$$

Definition 5.9. *We say that the quadruple $(\varphi,\ \sigma,\ \boldsymbol{u},\ z)$ is a weak solution to the approximate PDE system (5.1.26) endowed with the boundary and initial conditions (1.3.15)–(1.3.16) if*

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),$$

$$\sigma \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V),$$
$$\boldsymbol{u} \in W^{1,\infty}(0,T;V_0),$$
$$z \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),$$

with

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad z(0) = z_0 \quad a.e.\ in\ \Omega,$$

and it satisfies

$$\int_\Omega [\partial_t\varphi\zeta + \nabla\varphi\cdot\nabla\zeta]\,\mathrm{d}x = \int_\Omega [(p(\sigma,z)-\chi_1)\,\alpha(\varphi) - \varphi g(\sigma,z)]\,\zeta\,\mathrm{d}x, \tag{5.1.27a}$$

$$\langle\partial_t\sigma,\zeta\rangle_V + \int_\Omega \left[\nabla\sigma\cdot\nabla\zeta + \frac{k_1(\varphi,z)\sigma\zeta}{k_2(\varphi,z)+|\sigma|}\right]\mathrm{d}x + \int_\Gamma(\sigma-\sigma_\Gamma)\zeta\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \chi_2 S(\varphi,z)\zeta\,\mathrm{d}x, \tag{5.1.27b}$$

$$\int_\Omega [\mathcal{A}(\varphi,z)\varepsilon(\partial_t\boldsymbol{u}) + \mathcal{B}(\varphi,z)\varepsilon(\boldsymbol{u})] : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x, \tag{5.1.27c}$$

$$\int_\Omega [\partial_t z\zeta + \nabla z\cdot\nabla\zeta + \beta_\lambda(z)\zeta + \pi(z)\zeta]\,\mathrm{d}x = \int_\Omega [\iota - F(\varphi,\varepsilon(\boldsymbol{u}))]\,\zeta\,\mathrm{d}x, \tag{5.1.27d}$$

a.e. in $(0,T)$*, for every* $\zeta \in V$ *and* $\boldsymbol{v} \in V_0$*.*

Notice that, if $(\varphi,\sigma,\boldsymbol{u},z)$ is a solution of (5.1.26), (1.3.15), (1.3.16) in the sense of Definition 5.9 and we are able to prove that $0 \le \varphi \le N$ and $0 \le \sigma \le M$, then the truncation function in equation (5.1.26a) and the modulus in equation (5.1.26b) can be removed.

Proposition 5.10. *Assume that the set of hypotheses (A1)–(A9) holds. Then, for all* $\lambda \in (0,\lambda^*)$*, the approximate PDE system (5.1.26) endowed with the boundary and initial conditions (1.3.15)–(1.3.16) admits at least one weak solution* $(\varphi_\lambda,\sigma_\lambda,\boldsymbol{u}_\lambda,z_\lambda)$ *in the sense of Definition 5.9. Moreover, we have that*

$$0 \le \varphi_\lambda \le N, \qquad 0 \le \sigma_\lambda \le M \tag{5.1.28}$$

a.e. in Q*, where* $M = (M_0 + S^*)e^T$*. Finally, there exists a positive constant* C *depending only on the data of the problem and not depending on* λ *such that*

$$\|\varphi_\lambda\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \le C, \tag{5.1.29}$$
$$\|\sigma_\lambda\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} \le C, \tag{5.1.30}$$
$$\|\boldsymbol{u}_\lambda\|_{W^{1,\infty}(V_0)} \le C, \tag{5.1.31}$$
$$\|z_\lambda\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \le C, \tag{5.1.32}$$
$$\|\beta_\lambda(z_\lambda)\|_{L^2(H)} \le C. \tag{5.1.33}$$

Proof. The proof is based on the Schauder fixed-point argument (see e.g. [Bré11, p. 179]). We introduce the Banach space

$$\mathcal{X} \coloneqq \{(\sigma, z) \in L^2(0,T;H) \times L^2(0,T;L^\infty(\Omega))\}$$

endowed with the standard norm

$$\|(\sigma, z)\|_{\mathcal{X}} \coloneqq \|\sigma\|_{L^2(H)} + \|z\|_{L^2(L^\infty(\Omega))}.$$

In what follows, we construct an operator $\gamma : \mathcal{X} \to \mathcal{X}$, to which we intend to apply the Schauder fixed-point argument.

Step 1. Starting from $(\overline{\sigma}, \overline{z}) \in \mathcal{X}$, we find $\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ as the unique solution of the semilinear parabolic equation with Lipschitz continuous nonlinearity

$$\begin{cases} \partial_t \varphi - \Delta\varphi = (p(\overline{\sigma}, \overline{z}) - \chi_1)\, \alpha(\varphi) - \varphi g(\overline{\sigma}, \overline{z}) & \text{in } Q, \\ \partial_{\boldsymbol{\nu}} \varphi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_0 & \text{in } \Omega, \end{cases} \tag{5.1.34}$$

exploiting that $g(\overline{\sigma}, \overline{z}) \in L^\infty(Q)$ and $(p(\overline{\sigma}, \overline{z}) - \chi_1) \in L^\infty(Q)$. The well-posedness of this system can be proved in several classical ways, such as Galerkin discretization (see, e.g., [Trö10, Lemma 5.3, p. 373]) or semigroup theory (see, e.g., [Pat19, Chapter 20]). The regularity can be shown by standard results for linear parabolic equations (see [DL92; Lio61]), from which we obtain the estimate

$$\|\varphi\|_{H^1(H) \cap L^\infty(V) \cap L^2(W)} \le C, \tag{5.1.35}$$

for a certain positive constant C independent from λ and $(\overline{\sigma}, \overline{z})$. Next, we aim to prove that

$$0 \le \varphi \le N. \tag{5.1.36}$$

To do so, we test the first equation of system (5.1.34) with $(\varphi - N)^+$ and we integrate over Ω. Thanks to the boundary condition, we obtain:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |(\varphi - N)^+|^2 \,\mathrm{d}x &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |(\varphi - N)^+|^2 \,\mathrm{d}x + \int_\Omega |\nabla[(\varphi - N)^+]|^2 \,\mathrm{d}x \\ &= \int_\Omega (p(\overline{\varphi}, \overline{z}) - \chi_1)\, \alpha(\varphi)(\varphi - N)^+ \,\mathrm{d}x - \int_\Omega \varphi g(\overline{\sigma}, \overline{z})(\varphi - N)^+ \,\mathrm{d}x \le 0, \end{aligned}$$

where the last inequality holds because where $\varphi \ge N$ by definition $\alpha(\varphi) = 0$ (so the first addend is equal to 0) and φ is trivially positive (so the second addend is non-positive). Integrating in time, it follows that

$$\int_\Omega |(\varphi - N)^+|^2 \,\mathrm{d}x \le \int_\Omega |(\varphi_0 - N)^+|^2 \,\mathrm{d}x = 0,$$

employing hypothesis (A9) according to which $\varphi_0 \leq N$. As a consequence, $\varphi \leq N$ a.e. in Q. In a very similar way, we test the same equation by $-\varphi^-$ and integrate over Ω, obtaining:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi^-|^2 \,\mathrm{d}x \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi^-|^2 \,\mathrm{d}x + \int_\Omega |\nabla\varphi^-|^2 \,\mathrm{d}x$$
$$= -\int_\Omega (p(\overline{\sigma},\overline{z}) - \chi_1)\,\alpha(\varphi)\varphi^- \,\mathrm{d}x - \int_\Omega |\varphi^-|^2 g(\overline{\sigma},\overline{z})\,\mathrm{d}x \leq 0.$$

Integrating in time, it follows that

$$\int_\Omega |\varphi^-|^2 \,\mathrm{d}x \leq \int_\Omega |\varphi_0^-|^2 \,\mathrm{d}x = 0,$$

since $\varphi_0 \geq 0$ by hypothesis (A9), so $\varphi \geq 0$ a.e. in Q. Notice that we can consequently remove the truncation α.

Step 2. Starting from $(\overline{\sigma},\overline{z})$ and φ, we find $\sigma \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)$ as the unique solution of the following linear parabolic equation

$$\begin{cases} \partial_t\sigma - \Delta\sigma + \dfrac{k_1(\varphi,\overline{z})\sigma}{k_2(\varphi,\overline{z}) + |\overline{\sigma}|} = \chi_2 S(\varphi,\overline{z}) & \text{in } Q, \\ \partial_{\boldsymbol{\nu}}\sigma + \sigma - \sigma_\Gamma = 0 & \text{on } \Sigma, \\ \sigma(0) = \sigma_0 & \text{in } \Omega. \end{cases} \tag{5.1.37}$$

Using hypothesis (A2) and standard regularity results (see, e.g., [Lio61]), we also have that

$$\|\sigma\|_{H^1(V')\cap L^\infty(H)\cap L^2(V)} \leq C, \tag{5.1.38}$$

for a positive constant C that does not depend on λ, $(\overline{\sigma},\overline{z})$ and φ. Now we want to prove that there exists a positive constant M that depends only on M_0, R, S^*, and T such that

$$0 \leq \sigma \leq M \tag{5.1.39}$$

almost everywhere in Q. Multiplying the first equation in (5.1.37) with $-\sigma^-$, integrating over Ω and employing the boundary condition, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sigma^-|^2 \,\mathrm{d}x$$
$$\leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sigma^-|^2 \,\mathrm{d}x + \int_\Omega |\nabla\sigma^-|^2 \,\mathrm{d}x + \int_\Gamma |\sigma^-|^2 \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \frac{k_1(\varphi,\overline{z})|\sigma^-|^2}{k_2(\varphi,\overline{z}) + |\overline{\sigma}|}\,\mathrm{d}x$$
$$= -\int_\Omega \chi_2 S(\varphi,\overline{z})\sigma^- \,\mathrm{d}x - \int_\Gamma \sigma_\Gamma\sigma^- \,\mathrm{d}\mathcal{H}^{d-1} \leq 0.$$

Integrating in time over $(0,t)$, we have

$$\frac{1}{2}\int_\Omega |\sigma^-|^2 \,\mathrm{d}t \leq \frac{1}{2}\int_\Omega |\sigma_0^-|^2 \,\mathrm{d}t = 0,$$

where the last equality stands because of hypothesis (A9). As a consequence, $\sigma \geq 0$ a.e. in Q. Finally, we employ a standard Moser–Alikakos technique to prove that $\sigma \leq M$ for a certain $M > 0$ yet to be found. We multiply the first equation in (5.1.37) by $q\sigma^{q-1}$ with $q > 2$ and integrate in space over Ω. Notice that, in order to be sure that all the integrals are well-defined, one should introduce a truncation of σ,

$$\sigma_k := \begin{cases} \sigma & \text{if } \sigma \leq k, \\ k & \text{otherwise}, \end{cases}$$

for $k \in \mathbb{N}$, multiply the previous equation by $q(\sigma_k)^{q-1}$ and proceed as we will do. In the end, having obtained an estimate that does not depend on k, one should pass to the limit as $k \to +\infty$ and recover the thesis. We will not do it in this rigorous way to avoid overloading the exposition, and we proceed formally testing the first equation in (5.1.37) by $q\sigma^{q-1}$, obtaining

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sigma^q \,\mathrm{d}x + q(q-1)\int_\Omega \sigma^{q-2}|\nabla\sigma|^2 \,\mathrm{d}x + q\int_\Gamma \sigma^q \,\mathrm{d}\mathcal{H}^{d-1} + q\int_\Omega \frac{k_1(\varphi,\overline{z})\sigma^q}{k_2(\varphi,\overline{z}) + |\overline{\sigma}|}\,\mathrm{d}x \\
&\quad = q\int_\Gamma \sigma_\Gamma \sigma^{q-1} \,\mathrm{d}\mathcal{H}^{d-1} + q\int_\Omega \chi_2 S(\varphi,\overline{z})\sigma^{q-1}\,\mathrm{d}x \\
&\quad \leq qM_0\int_\Gamma \sigma^{q-1}\,\mathrm{d}\mathcal{H}^{d-1} + qRS^*\int_\Omega \sigma^{q-1}\,\mathrm{d}x \\
&\quad \leq (q-1)\int_\Gamma \sigma^q \,\mathrm{d}\mathcal{H}^{d-1} + (M_0)^q|\Gamma| + (q-1)\int_\Omega \sigma^q\,\mathrm{d}x + (RS^*)^q|\Omega|
\end{aligned}$$

using the boundedness of σ_Γ and S from hypotheses (A8) and (A2), together with the fact that $\chi_2 \in \mathcal{U}^2 = L^\infty(Q)$, and finally applying Young's inequality with exponents $q/(q-1)$ and q. Doing the obvious simplification in the previous inequality and employing the fact that σ is non-negative, we obtain:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sigma^q \,\mathrm{d}x \\
&\quad \leq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sigma^q \,\mathrm{d}x + q(q-1)\int_\Omega \sigma^{q-2}|\nabla\sigma|^2 \,\mathrm{d}x + \int_\Gamma \sigma^q \,\mathrm{d}\mathcal{H}^{d-1} + q\int_\Omega \frac{k_1(\varphi,\overline{z})\sigma^q}{k_2(\varphi,\overline{z}) + |\overline{\sigma}|}\,\mathrm{d}x \\
&\quad \leq (M_0)^q|\Gamma| + (q-1)\int_\Omega \sigma^q \,\mathrm{d}x + (RS^*)^q|\Omega|.
\end{aligned}$$

Integrating with respect to time over $(0,t)$, recalling that by hypothesis (A9) $\sigma_0 \leq M_0$, we find

$$\begin{aligned}
\int_\Omega \sigma^q(t)\,\mathrm{d}x &\leq \int_\Omega \sigma_0^q \,\mathrm{d}x + (M_0)^q|\Gamma|T + (RS^*)^q|\Omega|T + (q-1)\int_0^t\!\!\int_\Omega \sigma^q \,\mathrm{d}x\,\mathrm{d}t \\
&\leq (M_0)^q(|\Omega| + |\Gamma|T) + (RS^*)^q|\Omega|T + (q-1)\int_0^t\!\!\int_\Omega \sigma^q\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

whence we deduce

$$\|\sigma(t)\|_{L^q(\Omega)} \leq \left[M_0(|\Omega| + |\Gamma|T)^{\frac{1}{q}} + RS^*|\Omega|^{\frac{1}{q}}T^{\frac{1}{q}}\right] e^{\frac{q-1}{q}T}$$

using the Gronwall inequality and then taking the q^{th}-root of both sides of the inequality. Passing to the limit as $q \to +\infty$, we have

$$\|\sigma(t)\|_{L^\infty(\Omega)} \le M := (M_0 + RS^*)e^T$$

for a.e. $t \in (0,T)$.

Step 3. Starting from φ and $\overline{z}$, we find $\boldsymbol{u} \in W^{1,\infty}(0,T;V_0)$ as the unique solution of

$$\begin{cases} \displaystyle\int_\Omega [\mathcal{A}(\varphi,\overline{z})\varepsilon(\partial_t \boldsymbol{u}) + \mathcal{B}(\varphi,\overline{z})\varepsilon(\boldsymbol{u})] : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x & \forall \boldsymbol{v} \in V_0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{a.e. in } \Omega. \end{cases} \tag{5.1.40}$$

To do so, we proceed by time discretization. We consider a uniform partition of $[0,T]$ with time step $\tau > 0$ and equidistant nodes $0 = t_0 < t_1 < \cdots < t_{K_\tau} = T$. We also introduce the notation:

$$I_\tau^k := \begin{cases} [0,\tau] & \text{if } k = 1, \\ (t_\tau^{k-1},\ t_\tau^k] & \text{if } k = 2 \dots K_\tau. \end{cases}$$

We approximate φ, $\overline{z}$, and f with their local means, i.e., we define

$$f_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \boldsymbol{f}\,\mathrm{d}s, \qquad \varphi_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \varphi\,\mathrm{d}s, \qquad z_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \overline{z}\,\mathrm{d}s,$$

for every $k = 1,\dots,K_\tau$. To keep the notation short, we also introduce

$$\mathcal{A}_\tau^k := \mathcal{A}(\varphi_\tau^k, z_\tau^k), \qquad \mathcal{B}_\tau^k := \mathcal{B}(\varphi_\tau^k, z_\tau^k).$$

Remark 5.11. It is obvious that, since $\varphi, \overline{z} \in L^\infty(0,T;V)$ and $\boldsymbol{f} \in L^\infty(0,T;H)$, then $\varphi_\tau^k, z_\tau^k \in V$ and $f_\tau^k \in H$ with

$$\|\varphi_\tau^k\|_V \le \|\varphi\|_{L^\infty(V)}, \quad \|z_\tau^k\|_V \le \|\overline{z}\|_{L^\infty(V)}, \quad \|f_\tau^k\|_H \le \|f\|_{L^\infty(H)}, \tag{5.1.41}$$

for every $k = 1,\dots,K_\tau$.

Starting from $\boldsymbol{u}_\tau^0 = \boldsymbol{u}_0 \in V_0$, we solve recursively the following time-discrete problem:

$$\begin{cases} \text{Given } \boldsymbol{u}_\tau^{k-1} \in V_0, \text{ find } \boldsymbol{u}_\tau^k \in V_0 \text{ s.t. for all } \boldsymbol{v} \in V_0 \\ \displaystyle\int_\Omega \mathcal{A}_\tau^k \varepsilon\left(\frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}\right) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \mathcal{B}_\tau^k \varepsilon(\boldsymbol{u}_\tau^k) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega f_\tau^k \cdot \boldsymbol{v}\,\mathrm{d}x. \end{cases} \tag{5.1.42}$$

Since the equation in (5.1.42) can be rewritten as

$$\int_\Omega (\mathcal{A}_\tau^k + \tau\mathcal{B}_\tau^k)\varepsilon(\boldsymbol{u}_\tau^k) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \tau f_\tau^k \cdot \boldsymbol{v} + \mathcal{A}_\tau^k \varepsilon(\boldsymbol{u}_\tau^{k-1}) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x \tag{5.1.43}$$

where $\mathcal{A}_\tau^k + \tau\mathcal{B}_\tau^k$ is strictly positive definite and the right-hand side yields a linear bounded functional on V_0', the existence follows from Lax–Milgram Theorem. Now our aim is to pass to the limit in the discrete equation, recovering a solution to the original problem. For the sake of brevity, we introduce the shorter notation

$$\boldsymbol{v}_\tau^k := \frac{\boldsymbol{u}_\tau^k - \boldsymbol{u}_\tau^{k-1}}{\tau}$$

to denote the discrete velocity. Taking $\boldsymbol{v}_\tau^k$ as a test function in equation (5.1.42), it is easy to prove that

$$\|\boldsymbol{v}_\tau^k\|_{V_0} \leq C\left(\|\boldsymbol{f}\|_{L^\infty(H)} + \|\boldsymbol{u}_0\|_{V_0}\right), \tag{5.1.44}$$

where the constant C depends on T but non on k, λ, φ_τ^k, and z_τ^k. As a consequence, since

$$\boldsymbol{u}_\tau^k = \tau\sum_{j=1}^{k}\boldsymbol{v}_\tau^j + \boldsymbol{u}_0,$$

we also have that

$$\|\boldsymbol{u}_\tau^k\|_{V_0} \leq C \tag{5.1.45}$$

where, again C depends on $\boldsymbol{f}$, $\boldsymbol{u}_0$ and T but not on k, λ, φ_τ^k, and z_τ^k. Given any sequence of scalar, vector-valued or tensor-valued functions $\{w_\tau^k\}_{k=0}^{K_\tau}$ defined over Ω, we introduce the piecewise constant interpolations w_τ and the piecewise linear interpolation $\widehat{w}_\tau$ over the time interval $[0,T]$ as

$$w_\tau(t) := w_\tau^k, \qquad \widehat{w}_\tau(t) := \frac{t - t_\tau^{k-1}}{\tau}w_\tau^k + \frac{t_\tau^k - t}{\tau}w_\tau^{k-1}$$

for every $t \in I_\tau^k$. With this new notation, notice that $\partial_t\widehat{\boldsymbol{u}}_\tau = \boldsymbol{v}_\tau$ and estimates (5.1.44), (5.1.45) trivially lead to

$$\|\widehat{\boldsymbol{u}}_\tau\|_{W^{1,\infty}(V_0)} + \|\boldsymbol{u}_\tau\|_{L^\infty(V_0)} \leq C.$$

By standard compactness results, we obtain the existence of a function $\boldsymbol{u} \in W^{1,\infty}(0,T;V_0)$ such that

$$\begin{aligned} &\widehat{\boldsymbol{u}}_\tau \to \boldsymbol{u} \qquad \text{weakly-}* \qquad \text{in } W^{1,\infty}(0,T;V_0),\\ &\boldsymbol{u}_\tau \to \boldsymbol{u} \qquad \text{weakly-}* \qquad \text{in } L^\infty(0,T;V_0). \end{aligned}$$

Moreover, by their definition as local means in time, it holds true that

$$\boldsymbol{f}_\tau \to \boldsymbol{f} \qquad \text{strongly} \qquad \text{in } L^2(0,T;H)$$

and that

$$\varphi_\tau \to \varphi,\ z_\tau \to \overline{z} \qquad \text{a.e.} \qquad \text{in } Q.$$

Recalling that $\mathcal{A}$ and $\mathcal{B}$ are continuous and bounded by hypothesis (A3), it follows that

$$\mathcal{A}_\tau \to \mathcal{A}(\varphi,\overline{z}),\ \mathcal{B}_\tau \to \mathcal{B}(\varphi,\overline{z}) \qquad \text{strongly} \qquad \text{in } L^2(0,T;H)$$

by the Dominated Convergence Theorem. These convergences are enough to pass to the limit in the equation

$$\int_\Omega \mathcal{A}_\tau \varepsilon(\boldsymbol{v}_\tau) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \mathcal{B}_\tau \varepsilon(\boldsymbol{u}_\tau) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_\tau \cdot \boldsymbol{v}\,\mathrm{d}x$$

for every $\boldsymbol{v} \in V_0$ and a.e. $t \in (0,T)$, showing that $\boldsymbol{u}$ is a solution to the original system. Notice that from lower semi-continuity of the norm with respect to weak-$*$ convergence and estimate (5.1.3), we also have that

$$\|\boldsymbol{u}\|_{W^{1,\infty}(V_0)} \leq C, \tag{5.1.46}$$

where C depends on T, $\boldsymbol{f}$, and $\boldsymbol{u}_0$ but is independent of λ and $(\varphi, \overline{z})$. Finally, we should prove that the solution is unique, but this easily derives from the fact that the equation is linear in $\boldsymbol{u}$, $\mathcal{A}$ is strictly positive definite, and $\mathcal{B}$ is bounded.

Step 4. Starting from φ and $\boldsymbol{u}$, we find $z \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ as the unique solution of the semilinear parabolic equation with Lipschitz continuous nonlinearity

$$\begin{cases} \partial_t z - \Delta z + \beta_\lambda(z) + \pi(z) = \iota - F(\varphi, \varepsilon(\boldsymbol{u})) & \text{in } Q, \\ \partial_{\boldsymbol{\nu}} z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{5.1.47}$$

Notice that, thanks to hypothesis (A6), inequality (5.1.17), and estimates (5.1.35), (5.1.46), $\iota - F(\varphi, \varepsilon(\boldsymbol{u}))$ is uniformly bounded in $L^\infty(0,T;H)$. Now we want to prove that there exists a positive constant C (which does not depend on λ and $(\overline{\sigma}, \overline{z})$) such that

$$\|z\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \leq C. \tag{5.1.48}$$

Testing the first equation in (5.1.47) by $\partial_t z$ and integrating over Ω, we have

$$\begin{aligned}
&\int_\Omega |\partial_t z|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla z|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \widehat{\beta}_\lambda(z)\,\mathrm{d}x \\
&\quad = -\int_\Omega \pi(z)\partial_t z\,\mathrm{d}x + \int_\Omega [\iota - F(\varphi, \varepsilon(\boldsymbol{u}))]\,\partial_t z\,\mathrm{d}x \\
&\quad \leq C\int_\Omega \left(|z| + 1 + |\iota| + |\varphi| + |\varepsilon(\boldsymbol{u})| + |\hat{F}|\right)|\partial_t z|\,\mathrm{d}x \\
&\quad \leq \frac{1}{2}\int_\Omega |\partial_t z|^2\,\mathrm{d}x + C\int_\Omega \left(|z|^2 + 1 + |\iota|^2 + |\varphi|^2 + |\varepsilon(\boldsymbol{u})|^2 + |\hat{F}|^2\right)\,\mathrm{d}x \\
&\quad \leq \frac{1}{2}\int_\Omega |\partial_t z|^2\,\mathrm{d}x + C\int_\Omega |z_0|^2\,\mathrm{d}x + C\int_0^t\!\!\int_\Omega |\partial_t z|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + C\int_\Omega \left(1 + |\iota|^2 + |\varphi|^2 + |\varepsilon(\boldsymbol{u})|^2 + |\hat{F}|^2\right)\,\mathrm{d}x,
\end{aligned}$$

where we have used the fact that π is Lipschitz continuous by hypothesis (A5), inequality (5.1.17) (cf. hypothesis (A7)), applied the Young inequality and the fact that

$$z(t) = z_0 + \int_0^t \partial_t z \, \mathrm{d}s.$$

Integrating in time over $(0,\tau)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau\!\!\int_\Omega |\partial_t z|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla z|^2 \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\lambda(z)\,\mathrm{d}x \\
&\quad\le \frac{1}{2}\int_\Omega |\nabla z_0|^2 \,\mathrm{d}x + \int_\Omega \widehat{\beta}_\lambda(z_0)\,\mathrm{d}x + C\int_\Omega |z_0|^2\,\mathrm{d}x + C\int_0^\tau\!\!\int_0^t\!\!\int_\Omega |\partial_t z|^2\,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\
&\qquad + C\int_0^\tau\!\!\int_\Omega \left(1 + |\iota|^2 + |\varphi|^2 + |\varepsilon(\boldsymbol{u})|^2 + |\hat{F}|^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad\le C_0 + C + C\int_0^\tau\!\!\int_0^t\!\!\int_\Omega |\partial_t z|^2\,\mathrm{d}x\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

where we have used the fact that $\widehat{\beta}_\lambda \le \widehat{\beta}$ by definition of Yosida approximation (cf. Section 2.5.1). Applying the Gronwall inequality, we obtain

$$\|z\|_{H^1(H)\cap L^\infty(V)} \le C,$$

where $C > 0$ does not depend on $(\overline{\sigma}, \overline{z})$. By comparison in the first equation in (5.1.47), we have

$$\begin{aligned}
\|-\Delta z + \beta_\lambda(z)\|_{L^2(H)} &= \|-\partial_t z - \pi(z) + \iota - F(\varphi, \varepsilon(\boldsymbol{u}))\|_{L^2(H)} \\
&\le C\left(\|z\|_{H^1(H)} + 1 + \|\iota - F(\varphi,\varepsilon(\boldsymbol{u}))\|_{L^\infty(H)}\right) \le C
\end{aligned}$$

where C does not depend on λ and $(\overline{\sigma}, \overline{z})$. On the other hand, we observe that

$$\begin{aligned}
\|-\Delta z + \beta_\lambda(z)\|^2_{L^2(H)} &= \|-\Delta z\|^2_{L^2(H)} + \|\beta_\lambda(z)\|^2_{L^2(H)} + 2\int_0^t\!\!\int_\Omega -\Delta z\,\beta_\lambda(z)\,\mathrm{d}x\,\mathrm{d}s \\
&= \|-\Delta z\|^2_{L^2(H)} + \|\beta_\lambda(z)\|^2_{L^2(H)} + 2\int_0^t\!\!\int_\Omega \beta'_\lambda(z)|\nabla z|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\ge \|-\Delta z\|^2_{L^2(H)} + \|\beta_\lambda(z)\|^2_{L^2(H)},
\end{aligned}$$

where the inequality holds because β'_λ is monotone and Lipschitz continuous (so it is a.e. differentiable with non-negative derivative). Combining the previous two inequalities, we have proved

$$\|-\Delta z\|_{L^2(H)} + \|\beta_\lambda(z)\|_{L^2(H)} \le C$$

and, as a consequence, by elliptic regularity, we finally obtain estimate (5.1.48).

In the previous steps, we have built an operator $\gamma : \mathcal{X} \to \mathcal{X}$ such that $\gamma(\overline{\sigma}, \overline{z}) := (\sigma, z)$. From what we have already proved, it is straightforward that

$$\gamma \text{ is well-defined,}$$

because each of the problems (5.1.34), (5.1.37), (5.1.40), (5.1.47) is well-posed, and that

$$\gamma(\mathcal{X}) \text{ is a compact subset of } \mathcal{X},$$

because the following compact embeddings hold

$$\begin{aligned} H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V) &\hookrightarrow\hookrightarrow \; L^2(0,T;H), \\ H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) &\hookrightarrow\hookrightarrow \; L^2(0,T;L^\infty(\Omega)) \end{aligned}$$

by the Aubin–Lions Theorem (see Lemma 2.8). To apply the Schauder fixed point Theorem, it remains to prove that

$$\gamma \text{ is continuous with respect to the norm } \|\cdot\|_{\mathcal{X}}.$$

Thus, given a sequence $(\overline{\sigma}_k, \overline{z}_k)$ strongly converging to $(\overline{\sigma}, \overline{z})$ in $\mathcal{X}$, i.e. such that

$$\overline{\sigma}_k \to \overline{\sigma} \qquad \text{strongly} \qquad \text{in } L^2(0,T;H), \tag{5.1.49}$$

$$\overline{z}_k \to \overline{z} \qquad \text{strongly} \qquad \text{in } L^2(0,T;L^\infty(\Omega)), \tag{5.1.50}$$

we aim to verify that $(\sigma_k, z_k) \to (\sigma, z)$ strongly in $\mathcal{X}$. By the uniform estimates (5.1.38), (5.1.48) and standard compactness results, we know that there exists a pair (ρ, ζ) such that, along a subsequence that we do not relabel,

$$\sigma_k \to \rho \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.1.51}$$

$$\text{strongly} \qquad \text{in } L^2(0,T;H), \tag{5.1.52}$$

$$\text{a.e.} \qquad \text{in } Q, \tag{5.1.53}$$

$$z_k \to \zeta \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.1.54}$$

$$\text{strongly} \qquad \text{in } L^2(0,T;L^\infty(\Omega)), \tag{5.1.55}$$

$$\text{a.e.} \qquad \text{in } Q. \tag{5.1.56}$$

The proof is complete if we show that $(\rho, \zeta) = (\sigma, z)$, i.e. that ρ (resp. ζ) is the solution of problem (5.1.37) (resp. (5.1.47)) corresponding to the datum $(\overline{\sigma}, \overline{z})$. In fact, at this point, since every subsequence admits a sub-subsequence that converges to the same limit, the convergences (5.1.52) and (5.1.55) hold for the whole sequence. To do so, we pass to the limit as $k \to +\infty$ in the systems (5.1.34), (5.1.37), (5.1.40), (5.1.47) with initial datum $(\overline{\sigma}_k, \overline{z}_k)$ that, by definition of γ, are satisfied by φ_k, σ_k, $\boldsymbol{u}_k$ and z_k.

Step I. We know that φ_k satisfies

$$\partial_t \varphi_k - \Delta \varphi_k = (p(\overline{\sigma}_k, \overline{z}_k) - \chi_1)\, \alpha(\varphi_k) - \varphi_k g(\overline{\sigma}_k, \overline{z}_k)$$

in $L^2(0,T;H)$ and we aim to pass to the weak limit in this equation as $k \to +\infty$. From the uniform estimate (5.1.35) and standard compactness results, we assert that there exists a ϕ such that, along a further subsequence that we do not relabel,

$$\varphi_k \to \phi \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.1.57}$$

$$\begin{aligned} &\text{strongly} && \text{in } C^0([0,T];H^{1-\epsilon}(\Omega)) \cap L^2(0,T;H^{2-\epsilon}(\Omega)), && (5.1.58)\\ &\text{a.e} && \text{in } Q. && (5.1.59)\end{aligned}$$

for every $0 < \epsilon < 1$. Moreover, since $\overline{\sigma}_k \to \overline{\sigma}$ strongly in $L^2(0,T;H)$ and $\overline{z}_k \to \overline{z}$ strongly in $L^2(0,T;L^\infty(\Omega))$, it holds

$$\overline{\sigma}_k \to \overline{\sigma}, \ \overline{z}_k \to \overline{z} \qquad \text{a.e} \qquad \text{in } Q, \tag{5.1.60}$$

possibly extracting a further subsequence. This implies that we can pass to the limit in the equation: the terms on the left-hand side are trivial and the terms on the right-hand side converge strongly in $L^2(0,T;H)$ because we can apply the Lebesgue Convergence Theorem. In fact p, α and g are continuous and uniformly bounded (thanks to hypothesis (A1) and by definition of α in (5.1.25)), and their arguments converge a.e. (thanks to (5.1.59), (5.1.60)). So, ϕ actually solves system (5.1.34) with data $(\overline{\sigma}, \overline{z})$: by uniqueness of the solution, we may identify ϕ with φ.

Step II. The passage to the limit in the equation

$$\partial_t \sigma_k - \Delta \sigma_k + \frac{k_1(\varphi_k, \overline{z}_k)\sigma_k}{k_2(\varphi_k, \overline{z}_k) + |\overline{\sigma}_k|} = \chi_2 S(\varphi_k, \overline{z}_k)$$

is quite similar to the previous one, so we will not do it in detail. It allows us to assert that $\rho = \sigma$ and satisfies (5.1.37) for the data φ and $(\overline{\sigma}, \overline{z})$.

Step III. Regarding $\boldsymbol{u}_k$, the solution of (5.1.40) from the data $(\varphi_k, \overline{z}_k)$, there exists a $\boldsymbol{\omega}$ such that, along a non-relabeled subsequence,

$$\boldsymbol{u}_k \to \boldsymbol{\omega} \qquad \text{weakly-}* \qquad \text{in } W^{1,\infty}(0,T;V_0) \tag{5.1.61}$$

from (5.1.46) and standard compactness results. Thanks to this convergence, we are able to pass to the limit in the equation

$$\int_\Omega \mathcal{A}(\varphi_k, \overline{z}_k)\varepsilon(\partial_t \boldsymbol{u}_k) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \mathcal{B}(\varphi_k, \overline{z}_k)\varepsilon(\boldsymbol{u}_k) : \varepsilon(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x$$

for every fixed $\boldsymbol{v} \in V_0$. In fact, $\mathcal{A}$ and $\mathcal{B}$ are continuous from hypothesis (A3) and φ_k, $\overline{z}_k$ converge a.e. to φ, $\overline{z}$ employing (5.1.59) and (5.1.60), which implies that

$$\mathcal{A}(\varphi_k, \overline{z}_k) \to \mathcal{A}(\varphi, \overline{z}), \qquad \mathcal{B}(\varphi_k, \overline{z}_k) \to \mathcal{B}(\varphi, \overline{z})$$

a.e. in Q. Moreover, $\mathcal{A}$ and $\mathcal{B}$ are bounded by hypothesis (A3). Thus, (5.1.61) is enough to pass to the limit in the previous equation, and we may identify $\boldsymbol{\omega}$ with $\boldsymbol{u}$ since it is the unique solution of system (5.1.40) with initial data φ and $\overline{z}$.

Step IV. The passage to the limit in the equation

$$\partial_t z_k - \Delta z_k + \beta_\lambda(z_k) + \pi(z_k) = \iota - F(\varphi_k, \varepsilon(\boldsymbol{u}_k)),$$

is obvious in all the terms with the exception of $F(\varphi_k, \varepsilon(\boldsymbol{u}_k))$, because of the convergence (5.1.54), (5.1.55) and the Lipschitz continuity of β_λ and π. However, we need to prove stronger convergence for $\varepsilon(\boldsymbol{u}_k)$ to treat the last one. Thus, we take the difference of the equations satisfied by $\boldsymbol{u}_k$ and $\boldsymbol{u}_l$ and test it with $\partial_t \boldsymbol{v}$, where $\boldsymbol{v} := \boldsymbol{u}_k - \boldsymbol{u}_l$. After summing and subtracting some terms, we obtain:

$$\begin{aligned}\int_\Omega \mathcal{A}(\varphi_k, \overline{z}_k)\varepsilon(\partial_t \boldsymbol{v}) : \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}x &= -\int_\Omega \left[\mathcal{A}(\varphi_k, \overline{z}_k) - \mathcal{A}(\varphi_l, \overline{z}_l)\right] \varepsilon(\partial_t \boldsymbol{u}_l) : \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}x \\ - \int_\Omega \mathcal{B}(\varphi_k, \overline{z}_k)\varepsilon(\boldsymbol{v}) : \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}x &- \int_\Omega \left[\mathcal{B}(\varphi_k, \overline{z}_k) - \mathcal{B}(\varphi_l, \overline{z}_l)\right] \varepsilon(\boldsymbol{u}_l) : \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}x.\end{aligned}$$

Exploiting strictly positive definiteness of $\mathcal{A}$, Lipschitz continuity of $\mathcal{A}$ and $\mathcal{B}$, boundedness of $\mathcal{B}$ from hypothesis (A3), and using the Hölder and the Young inequalities, for a.e. $t \in (0, T)$ we get

$$\begin{aligned}C_{\mathcal{A}_*} \int_\Omega |\varepsilon(\partial_t \boldsymbol{v})|^2\,\mathrm{d}x &\le \eta \int_\Omega |\varepsilon(\partial_t \boldsymbol{v})|^2\,\mathrm{d}x + C_\eta \int_\Omega |\varepsilon(\boldsymbol{v})|^2\,\mathrm{d}x \\ &+ C_\eta \left(\|\varphi_k - \varphi_l\|^2_{L^\infty(\Omega)} + \|\overline{z}_k - \overline{z}_l\|^2_{L^\infty(\Omega)}\right) \int_\Omega \left(|\varepsilon(\partial_t \boldsymbol{u}_l)|^2 + |\varepsilon(\boldsymbol{u}_l)|^2\right)\,\mathrm{d}x\end{aligned}$$

for a positive η small enough. Then we integrate in time over the interval $(0, t)$ and move to the left-hand side the term multiplied by the small coefficient η. Moreover, from (5.1.46), we know that $\|\varepsilon(\boldsymbol{u}_l)\|_H$ and $\|\varepsilon(\partial_t \boldsymbol{u}_l)\|_H$ are uniformly bounded in time and the following equality holds

$$\varepsilon(\boldsymbol{v}(s)) = \varepsilon(\boldsymbol{v}(0)) + \int_0^s \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}\tau = \int_0^s \varepsilon(\partial_t \boldsymbol{v})\,\mathrm{d}\tau,$$

where we have used the fact that $\boldsymbol{u}_k(0) = \boldsymbol{u}_l(0) = \boldsymbol{u}_0$. Combining all these elements, the previous inequality becomes

$$\begin{aligned}\int_0^t\!\!\int_\Omega |\varepsilon(\partial_t \boldsymbol{v})|^2\,\mathrm{d}s \le C\Bigg[&\int_0^t \left(\int_0^s\!\!\int_\Omega |\varepsilon(\partial_t \boldsymbol{v})|^2\,\mathrm{d}x\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \|\varphi_k - \varphi_l\|^2_{L^2(L^\infty(\Omega))} + \|\overline{z}_k - \overline{z}_l\|^2_{L^2(L^\infty(\Omega))}\Bigg].\end{aligned}$$

Applying the Gronwall inequality, we obtain

$$\|\varepsilon(\partial_t \boldsymbol{v})\|^2_{L^2(H)} \le Ce^{CT}\left[\|\varphi_k - \varphi_l\|^2_{L^2(L^\infty(\Omega))} + \|\overline{z}_k - \overline{z}_l\|^2_{L^2(L^\infty(\Omega))}\right] \to 0$$

as $k, l \to +\infty$ thanks to the strong convergences (5.1.55) and (5.1.58) with ϵ small enough (so that the embedding $H^{2-\epsilon} \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$ holds). This implies that also $\|\varepsilon(\boldsymbol{v})\|_{L^\infty(H)}$ vanishes in the limit. So, $\{\boldsymbol{u}_k\}$ is a Cauchy sequence in $H^1(0, T; V_0)$ and consequently converges. Thus, we have proved that

$$\boldsymbol{u}_k \to \boldsymbol{u} \qquad \text{strongly} \qquad \text{in } H^1(0, T; V_0), \tag{5.1.62}$$

where we are able to identify the limit with $\boldsymbol{u}$ because of the known convergence (5.1.61). Now we can conclude the passage to the limit in the damage equation. In fact, F is Lipschitz continuous by hypothesis (A7), $\varphi_k \to \varphi$ strongly in $L^2(0,T;H)$ by (5.1.58) and $\varepsilon(\boldsymbol{u}_k) \to \varepsilon(\boldsymbol{u})$ strongly in $L^2(0,T;H)$ by (5.1.62), so $F(\varphi_k, \varepsilon(\boldsymbol{u}_k)) \to F(\varphi, \varepsilon(\boldsymbol{u}))$ strongly in $L^2(0,T;H)$. Finally, since ζ satisfies (5.1.47) with input data φ and $\boldsymbol{u}$, we can deduce the identification $\zeta = z$.

Applying the Schauder fixed point Theorem, it follows that there exists a pair $(\sigma_\lambda, z_\lambda)$ such that $(\sigma_\lambda, z_\lambda) = \gamma(\sigma_\lambda, z_\lambda)$. By construction of γ, we have proved the existence of a quadruple $(\varphi_\lambda, \sigma_\lambda, \boldsymbol{u}_\lambda, z_\lambda)$ that is a weak solution of the approximate problem in the sense of Definition 5.9. Moreover, this solution satisfies the uniform estimates we obtained throughout the proof since they did not depend on λ, so (5.1.29)–(5.1.33) hold. □

Conclusion of the proof of Theorem 5.5. Let us consider a sequence of weak solutions of the approximate problem $\{(\varphi_\lambda, \sigma_\lambda, \boldsymbol{u}_\lambda, z_\lambda)\}_\lambda$. Now we want to pass to the limit as $\lambda \to 0$. Employing the uniform estimates (5.1.29), (5.1.30), (5.1.31), (5.1.32), (5.1.33), there exist a quadruple $(\varphi, \sigma, \boldsymbol{u}, z)$ and a ξ such that

$$\varphi_\lambda \to \varphi \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.1.63}$$

$$\sigma_\lambda \to \sigma \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.1.64}$$

$$\boldsymbol{u}_\lambda \to \boldsymbol{u} \qquad \text{weakly-}* \qquad \text{in } W^{1,\infty}(0,T;V_0), \tag{5.1.65}$$

$$z_\lambda \to z \qquad \text{weakly-}* \qquad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.1.66}$$

$$\beta_\lambda(z_\lambda) \to \xi \qquad \text{weakly} \qquad \text{in } L^2(0,T;H) \tag{5.1.67}$$

along a subsequence of λ that we do not relabel. The passage to the limit in the approximate system can be performed exactly as in the proof of Proposition 5.10, proving strong convergence where needed through standard compactness results and directly for the displacement equation. The only difference that needs further discussion is the Yosida approximation in the damage equation, because we need to justify that $\xi \in \beta(z)$. From (5.1.32) and the Aubin–Lions compact embedding

$$H^1(0,T;H) \cap L^2(0,T;W) \hookrightarrow\hookrightarrow L^2(0,T;V),$$

we gather that $z_\lambda \to z$ strongly in $L^2(0,T;H)$ along a further non-relabeled subsequence. Thus, since β is maximal monotone and β_λ is its Yosida approximation, we exploit Proposition 2.14 and deduce that $\xi \in \beta(z)$ a.e. in Q. This concludes the proof of Theorem 5.5.

5.1.4 Regularity

As we will comment further later on, the prescribed regularity can be proved on the approximate level using standard regularity results. Then, the only thing to be shown is that such regularity passes to the limit, and this will be done by proving some a priori estimates in the stronger norms we need.

Lemma 5.12. *Under hypotheses (A1)–(A9)–(B1)–(B4), the solution to the approximate problem found in Proposition 5.10 enjoys the further regularity*

$$\varphi_\lambda \in H^1(0,T;V) \cap L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega)),$$
$$\boldsymbol{u}_\lambda \in W^{1,\infty}(0,T;W_0),$$
$$z_\lambda \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W).$$

Moreover, it satisfies

$$\|\varphi_\lambda\|_{H^1(V)\cap L^\infty(W)\cap L^2(H^3(\Omega))} \le C, \tag{5.1.68}$$
$$\|\boldsymbol{u}_\lambda\|_{W^{1,\infty}(W_0)} \le C, \tag{5.1.69}$$
$$\|z_\lambda\|_{H^1(V)\cap L^\infty(W)} \le C, \tag{5.1.70}$$
$$\|\beta_\lambda(z_\lambda)\|_{L^\infty(H)} \le C, \tag{5.1.71}$$

for a positive constant C depending only on the problem data and not on λ.

Proof. To simplify the notation, since here λ is fixed, we will omit it.

Estimate for φ. The right-hand side of equation (5.1.26a) is

$$h \coloneqq (p(\sigma,z) - \chi_1)\,\varphi\left(1 - \frac{\varphi}{N}\right) - \varphi g(\sigma,z).$$

Notice that we have removed the truncation α because we have already proved that $0 \le \varphi \le N$ (cf. (5.1.28) in Proposition 5.10). From well-known regularity theory for parabolic equations (see [DL92; Lio61]), since $h \in L^2(0,T;V)$, φ enjoys the regularity declared in the statement and it holds

$$\|\varphi\|_{H^1(V)\cap L^\infty(W)\cap L^2(H^3(\Omega))} \le C(\|h\|_{L^2(V)} + \|\varphi_0\|_W). \tag{5.1.72}$$

We aim to bound $\|h\|_{L^2(V)}$. Since p, g are bounded by hypothesis (A1), and $\chi_1 \in \mathcal{U}^1_{\mathrm{ad}} \subseteq L^\infty(Q)$, it follows that

$$\|h\|_{L^2(H)} \le (p^* + R + g^*)N|\Omega|^{\frac{1}{2}}.$$

Moreover, we have that

$$\nabla h = (p_{,\sigma}\nabla\sigma + p_{,z}\nabla z - \nabla\chi_1)\varphi\left(1 - \frac{\varphi}{N}\right) + (p(\sigma,z) - \chi_1)\left[\nabla\varphi\left(1 - \frac{\varphi}{N}\right) - \varphi\frac{\nabla\varphi}{N}\right]$$
$$- g(\sigma,z)\nabla\varphi - (g_{,\sigma}\nabla\sigma + g_{,z}\nabla z)\varphi.$$

Recalling that φ and z are bounded, p, g are bounded along with their partial derivatives by hypotheses (A1), (B1), and that $\chi_1 \in \mathcal{U}^1_{\mathrm{ad}}$, we have

$$|\nabla h| \le C(|\nabla\sigma| + |\nabla z| + |\nabla\chi_1| + |\nabla\varphi|).$$

Thus, since we have already proved (5.1.30), (5.1.32), (5.1.29), we get that $\|\nabla h\|_{L^2(H)}$ is uniformly bounded. Hence, from (5.1.72), we obtain (5.1.68).

Estimate for z and $\beta_\lambda(z)$. Observing that the term

$$\iota - F(\varphi, \varepsilon(\boldsymbol{u})) - \beta_\lambda(z) - \pi(z)$$

belongs to $H^1(0,T;H)$, by standard parabolic regularity results (see [DL92; Lio61]), z has the desired regularity. Unfortunately, estimate (5.1.70) (which is independent of λ) cannot be deduced as we have done for φ because the Lipschitz constant of the Yosida approximation β_λ depends on λ. To overcome this difficulty, we aim to test equation (5.1.26d) by $\partial_t(-\Delta z + \beta_\lambda(z))$ and integrate over the time interval $(0,t)$. Notice that the following calculations are formal, since $\partial_t(-\Delta z + \beta_\lambda(z))$ does not possess the regularity $L^2(0,T;H)$, which would be suitable for Equation (5.1.26d). However, the same estimate can be performed rigorously at the discrete level in a Galerkin scheme, so we will not enter into technical details. We have:

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega |\nabla(\partial_t z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \beta_\lambda'(z)|\partial_t z|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \frac{1}{2}\left[\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x - \int_\Omega |-\Delta z_0 + \beta_\lambda(z_0)|^2 \,\mathrm{d}x\right] \\
&= -\int_0^t\!\!\int_\Omega (\pi(z) + F(\varphi, \varepsilon(\boldsymbol{u})) - \iota)\partial_t(-\Delta z + \beta_\lambda(z)) \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Then we integrate by parts the term on the right-hand side with respect to time, obtaining:

$$\begin{aligned}
&\int_0^t \int_\Omega |\nabla(\partial_t z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \beta_\lambda'(z)|\partial_t z|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x \\
&= \int_\Omega (\pi(z_0) + F(\varphi_0, \varepsilon(\boldsymbol{u}_0)) - \iota(0))(-\Delta z_0 + \beta_\lambda(z_0)) \,\mathrm{d}x \\
&\quad - \int_\Omega (\pi(z) + F(\varphi, \varepsilon(\boldsymbol{u})) - \iota)(-\Delta z + \beta_\lambda(z)) \,\mathrm{d}x \\
&\quad + \int_0^t\!\!\int_\Omega \left(\pi'(z)\partial_t z + \partial_t F(\varphi, \varepsilon(\boldsymbol{u})) - \partial_t \iota\right)(-\Delta z + \beta_\lambda(z)) \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \frac{1}{2}\int_\Omega |-\Delta z_0 + \beta_\lambda(z_0)|^2 \,\mathrm{d}x \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.1.73}$$

Regarding the left-hand side, since β_λ' is non negative because β_λ is monotone, we have

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega |\nabla(\partial_t z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x \\
&\quad \leq \int_0^t\!\!\int_\Omega |\nabla(\partial_t z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_\Omega \beta_\lambda'(z)|\partial_t z|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\quad + \frac{1}{2}\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x.
\end{aligned} \tag{5.1.74}$$

Next, we aim to bound the terms on the right-hand side. As regards I_1 and I_4, which depend on the initial data, we only observe that, by well-known properties of the Yosida approximation (see Section 2.5.1),

$$|\beta_\lambda(z_0)| \leq |\beta^0(z_0)|,$$

where the right-hand side is bounded because of hypothesis (B4). All the other addends in I_1 and I_4 do not depend on λ and are bounded because of the hypotheses we imposed on the initial data. Concerning I_2, we recall that π and F are Lipschitz continuous by hypotheses (A5) and (A7). Then we apply the Young inequality. We get

$$\begin{aligned} I_2 &\leq C \int_\Omega (|z| + 1 + |\varepsilon(\boldsymbol{u})| + |\varphi| + |\iota|)\, |-\Delta z + \beta_\lambda(z)| \,\mathrm{d}x \\ &\leq \frac{1}{4} \int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x + C \int_\Omega \left(|z|^2 + 1 + |\varepsilon(\boldsymbol{u})|^2 + |\varphi|^2 + |\iota|^2\right) \mathrm{d}x \\ &\leq \frac{1}{4} \int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x + C, \end{aligned} \tag{5.1.75}$$

where the last inequality is due to hypothesis (A6) and to the fact that z, $\varepsilon(\boldsymbol{u})$, and φ are uniformly bounded in $L^\infty(0,T;H)$ by (5.1.29), (5.1.31) and (5.1.32). The term I_3 can be handled similarly, by using in particular (A7). We have

$$\begin{aligned} I_3 &\leq C \int_0^t\!\!\int_\Omega (|\partial_t z| + |\varepsilon(\partial_t \boldsymbol{u})| + |\partial_t \varphi| + |\partial_t \iota|)\, |-\Delta z + \beta_\lambda(z)| \,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq C \int_0^t\!\!\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + C \int_0^t\!\!\int_\Omega \left(|\partial_t z|^2 + |\varepsilon(\partial_t \boldsymbol{u})|^2 + |\partial_t \varphi|^2 + |\partial_t \iota|^2\right) \mathrm{d}x\,\mathrm{d}\tau \\ &\leq C \int_0^t\!\!\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + C, \end{aligned} \tag{5.1.76}$$

recalling hypothesis (B3) and the fact that $\partial_t z$, $\varepsilon(\partial_t \boldsymbol{u})$ and $\partial_t \varphi$ are uniformly bounded in $L^2(0,T;H)$ again by (5.1.29), (5.1.31), (5.1.32). Combining (5.1.73), (5.1.74), (5.1.75), (5.1.76) leads to

$$\int_0^t\!\!\int_\Omega |\nabla(\partial_t z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{4}\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x \leq C + C \int_0^t\!\!\int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x\,\mathrm{d}\tau.$$

Thus, applying the Gronwall Lemma, we conclude that the left-hand side is uniformly bounded. Moreover, since β'_λ is non-negative, it holds

$$\begin{aligned} C &\geq \int_\Omega |-\Delta z + \beta_\lambda(z)|^2 \,\mathrm{d}x = \int_\Omega |-\Delta z|^2 + |\beta_\lambda(z)|^2 - 2\Delta z \beta_\lambda(z) \,\mathrm{d}x \\ &= \int_\Omega |-\Delta z|^2 + |\beta_\lambda(z)|^2 + 2\beta'_\lambda(z)|\nabla z|^2 \,\mathrm{d}x \geq \int_\Omega |-\Delta z|^2 + |\beta_\lambda(z)|^2 \,\mathrm{d}x. \end{aligned}$$

and, consequently, estimates (5.1.71) and (5.1.70) are satisfied.

Estimate for $\boldsymbol{u}$. The additional regularity for $\boldsymbol{u}$ can be proved at the time-discrete level, noticing that, starting from $\boldsymbol{u}_0 \in W_0$, the right-hand side in equation (5.1.43) can be represented as a linear functional on H. As a consequence, applying the regularity result [MH94, Theorem 1.11, p. 322], the time-discrete solution is in W. To prove that the limit solution preserves this regularity, one should prove an estimate uniform with respect to the time-step τ. Anyway, we are going to omit these calculations since they are similar to the ones we are going to perform in the continuous setting, where we search for an estimate uniform in λ.

Since $\boldsymbol{u} \in W^{1,\infty}(0,T;W_0)$, equation (5.1.26c) can be rewritten as

$$\begin{aligned}-\mathcal{A}(\varphi,z)\,\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})] =& \mathcal{B}(\varphi,z)\,\mathrm{div}[\varepsilon(\boldsymbol{u})] + \varepsilon(\partial_t\boldsymbol{u})\,[\mathcal{A}_{,\varphi}(\varphi,z)\nabla\varphi + \mathcal{A}_{,z}(\varphi,z)\nabla z]\\ &+ \varepsilon(\boldsymbol{u})\,[\mathcal{B}_{,\varphi}(\varphi,z)\nabla\varphi + \mathcal{B}_{,z}(\varphi,z)\nabla z] + \boldsymbol{f}\end{aligned}$$

which is satisfied in H for a.e. $t \in (0,T)$. Multiplying it by $-\,\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]$ and integrating over Ω, one gets

$$\begin{aligned}\int_\Omega \mathcal{A}(\varphi,z)\,\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x =& -\int_\Omega \mathcal{B}(\varphi,z)\,\mathrm{div}[\varepsilon(\boldsymbol{u})]\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x\\ &-\int_\Omega \varepsilon(\partial_t\boldsymbol{u})\,[\mathcal{A}_{,\varphi}(\varphi,z)\nabla\varphi + \mathcal{A}_{,z}(\varphi,z)\nabla z]\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x\\ &-\int_\Omega \varepsilon(\boldsymbol{u})\,[\mathcal{B}_{,\varphi}(\varphi,z)\nabla\varphi + \mathcal{B}_{,z}(\varphi,z)\nabla z]\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x\\ &-\int_\Omega \boldsymbol{f}\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x =: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{5.1.77}$$

Recalling that, by hypothesis (A3), $\mathcal{A}$ is strictly positive definite,

$$C_{\mathcal{A}_*}\|\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\|_H^2 \le \int_\Omega \mathcal{A}(\varphi,z)\,\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\cdot\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}x. \tag{5.1.78}$$

Now we are going to estimate the terms on the right-hand side of equation (5.1.77). Starting from I_1, we employ the boundedness of $\mathcal{B}$ from hypothesis (A3), the Young inequality with a small parameter η yet to be determined, and the following equality

$$\mathrm{div}[\varepsilon(\boldsymbol{u})](\tau) = \mathrm{div}[\varepsilon(\boldsymbol{u}_0)] + \int_0^\tau \mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\,\mathrm{d}t$$

and we obtain

$$\begin{aligned}I_1 &\le \eta\|\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\|_H^2 + C_\eta\|\mathrm{div}[\varepsilon(\boldsymbol{u})]\|_H^2\\ &\le \eta\|\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\|_H^2 + C_\eta\left(\|\mathrm{div}[\varepsilon(\boldsymbol{u}_0)]\|_H^2 + \int_0^\tau \|\mathrm{div}[\varepsilon(\partial_t\boldsymbol{u})]\|_H^2\,\mathrm{d}t\right),\end{aligned} \tag{5.1.79}$$

where the constant C_η has changed in the passage from the first to the second line.

Regarding I_2, by Lipschitz continuity of $\mathcal{A}$ and $\mathcal{B}$, the Hölder inequality and the Young inequality with a small η, we get

$$I_2 \leq \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\eta \|\varepsilon(\partial_t \boldsymbol{u})\|_{L^4(\Omega)}^2 \left(\|\nabla\varphi\|_{L^4(\Omega)}^2 + \|\nabla z\|_{L^4(\Omega)}^2\right). \tag{5.1.80}$$

Since we have already shown that $\|\varphi\|_W, \|z\|_W \leq C$ from (5.1.68), (5.1.70), by the continuous embedding $W \hookrightarrow W^{1,4}(\Omega)$ it follows

$$\|\nabla\varphi\|_{L^4(\Omega)}^2 + \|\nabla z\|_{L^4(\Omega)}^2 \leq C. \tag{5.1.81}$$

Employing Ehrling's Lemma with $W \hookrightarrow\hookrightarrow W^{1,4}(\Omega) \hookrightarrow H$ and a small constant θ yet to be determined, it holds

$$\begin{aligned}\|\varepsilon(\partial_t \boldsymbol{u})\|_{L^4(\Omega)}^2 &\leq C\|\partial_t \boldsymbol{u}\|_{W^{1,4}(\Omega)}^2 \leq \theta\|\partial_t \boldsymbol{u}\|_W^2 + C_\theta\|\partial_t \boldsymbol{u}\|_H^2 \\ &\leq \theta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\theta.\end{aligned}$$

The last inequality follows from Lemma 2.23 and the previous estimate (5.1.31), renaming the constant involved. Combining these elements, starting from (5.1.80) we have proved that

$$I_2 \leq (\eta+\theta)\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_{\eta,\theta}. \tag{5.1.82}$$

The term I_3 can be treated similarly. Proceeding as in equation (5.1.80) and taking into account equation (5.1.81), yields

$$I_3 \leq \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\eta\|\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)}^2 \leq \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\eta\|\boldsymbol{u}\|_W^2.$$

Since

$$\boldsymbol{u}(\tau) = \boldsymbol{u}_0 + \int_0^\tau \partial_t \boldsymbol{u}\,\mathrm{d}t,$$

we have

$$\begin{aligned}I_3 &\leq \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\eta\left(\|\boldsymbol{u}_0\|_W^2 + \int_0^\tau \|\partial_t \boldsymbol{u}\|_W^2\,\mathrm{d}t\right) \\ &\leq \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2 + C_\eta\left(1 + \int_0^\tau \|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2\,\mathrm{d}t\right),\end{aligned} \tag{5.1.83}$$

where the last inequality follows from Lemma 2.23, changing the constant C_η. Finally, by the Young and the Hölder inequalities,

$$I_4 \leq \|\boldsymbol{f}\|_H\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H \leq C_\eta\|\boldsymbol{f}\|_H^2 + \eta\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2. \tag{5.1.84}$$

Now we combine (5.1.77) with (5.1.78), (5.1.79), (5.1.82), (5.1.83), (5.1.84) and we move to the left-hand side the terms with η and θ, fixing them small enough. We obtain:

$$\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2(\tau) \leq C\left(1 + \|\boldsymbol{f}\|_{L^\infty(H)}^2 + \int_0^\tau \|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_H^2\,\mathrm{d}t\right).$$

Applying Gronwall's inequality, it follows that $\|\mathrm{div}[\varepsilon(\partial_t \boldsymbol{u})]\|_{L^\infty(H)}$ is uniformly bounded, and, thanks to Lemma 2.23, that $\|\partial_t \boldsymbol{u}\|_{L^\infty(W)}$ is in turn uniformly bounded. Thus, (5.1.69) holds. This concludes the proof of Theorem 5.7. □

5.1.5 Continuous dependence

Consider two pairs $\{(\varphi_i, \sigma_i, \boldsymbol{u}_i, z_i)\}_{i=1,2}$ of weak solutions corresponding to the assigned functions $\{(\boldsymbol{\chi}_i, \boldsymbol{f}_i, w_i, \sigma_{\Gamma,i})\}_{i=1,2}$ and to the initial data $\{(\varphi_{0,i}, \sigma_{0,i}, \boldsymbol{u}_{0,i}, z_{0,i})\}_{i=1,2}$. For the sake of brevity, in the following, we will use the shorter notation

$$\begin{aligned}
&\varphi := \varphi_1 - \varphi_2, && \sigma := \sigma_1 - \sigma_2, && \boldsymbol{u} := \boldsymbol{u}_1 - \boldsymbol{u}_2, && z := z_1 - z_2,\\
&\varphi_0 := \varphi_{0,1} - \varphi_{0,2}, && \sigma_0 := \sigma_{0,1} - \sigma_{0,2}, && \boldsymbol{u}_0 := \boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}, && z_0 := z_{0,1} - z_{0,2},\\
&\boldsymbol{\chi} := \boldsymbol{\chi}_1 - \boldsymbol{\chi}_2, && \boldsymbol{f} := \boldsymbol{f}_1 - \boldsymbol{f}_2, && \iota := \iota_1 - \iota_2, && \sigma_\Gamma := \sigma_{\Gamma,1} - \sigma_{\Gamma,2},
\end{aligned}$$

and we will denote $M := \max_{i=1,2} M_i$. We specify that we write each vector $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2$ as follows

$$\boldsymbol{\chi}_1 = ((\chi_1)_1, (\chi_1)_2), \qquad \boldsymbol{\chi}_2 = ((\chi_2)_1, (\chi_2)_2).$$

Thus, $\boldsymbol{\chi} = (\chi_1, \chi_2)$ with

$$\chi_1 = (\chi_1)_1 - (\chi_2)_1, \qquad \chi_2 = (\chi_1)_2 - (\chi_2)_2.$$

Taking the difference of (5.1.23a) written for φ_1 and φ_2, testing it by φ, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi|^2\,\mathrm{d}x + \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x = -\int_\Omega \left(g(\sigma_1, z_1) - g(\sigma_2, z_2)\right)\varphi_1\varphi\,\mathrm{d}x\\
&\quad - \int_\Omega g(\sigma_2, z_2)\varphi^2\,\mathrm{d}x + \int_\Omega \left(p(\sigma_1, z_1) - p(\sigma_2, z_2) - \chi_1\right)\varphi_1\Big(1 - \frac{\varphi_1}{N}\Big)\varphi\,\mathrm{d}x\\
&\quad + \int_\Omega \left(p(\sigma_2, z_2) - (\chi_2)_2\right)\Big[\varphi_1\Big(1 - \frac{\varphi_1}{N}\Big) - \varphi_2\Big(1 - \frac{\varphi_2}{N}\Big)\Big]\varphi\,\mathrm{d}x \qquad (5.1.85)\\
&\quad \le C\Big[\int_\Omega (|\sigma| + |z| + |\chi_1|)|\varphi|\,\mathrm{d}x + \int_\Omega |\varphi|^2\,\mathrm{d}x\Big]\\
&\quad \le C\Big[\int_\Omega \left(|\sigma|^2 + |z|^2 + |\chi_1|^2 + |\varphi|^2\right)\,\mathrm{d}x\Big],
\end{aligned}$$

where the inequality follows from the Young inequality and the constant C depends on N, g^*, p^*, R, and the Lipschitz constants of p and g. Taking the difference of (5.1.23b) written for σ_1 and σ_2, testing it by σ, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |\nabla\sigma|^2\,\mathrm{d}x + \int_\Gamma |\sigma|^2\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \frac{k_1(\varphi_1, z_1)}{k_2(\varphi_1, z_1) + \sigma_1}\sigma^2\,\mathrm{d}x\\
&\quad = \int_\Gamma \sigma_\Gamma\sigma\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \frac{k_1(\varphi_1, z_1) - k_1(\varphi_2, z_2)}{k_2(\varphi_1, z_1) + \sigma_1}\sigma_2\sigma\,\mathrm{d}x\\
&\quad - \int_\Omega \left(\frac{k_1(\varphi_2, z_2)}{k_2(\varphi_1, z_1) + \sigma_1} - \frac{k_1(\varphi_2, z_2)}{k_2(\varphi_2, z_2) + \sigma_2}\right)\sigma_2\sigma\,\mathrm{d}x\\
&\quad + \int_\Omega \left[(\chi_1)_2\left(S(\varphi_1, z_1) - S(\varphi_2, z_2)\right) + \chi_2 S(\varphi_2, z_2)\right]\sigma\,\mathrm{d}x\\
&\quad \le \frac{1}{2}\int_\Gamma |\sigma_\Gamma|^2\,\mathrm{d}\mathcal{H}^{d-1} + \frac{1}{2}\int_\Gamma |\sigma|^2\,\mathrm{d}\mathcal{H}^{d-1} + C\int_\Omega (|\varphi| + |z| + |\chi_2| + |\sigma|)|\sigma|\,\mathrm{d}x
\end{aligned}$$

where we have used hypotheses (A2) and (B2) for k_1, k_2 and S, the fact that $\boldsymbol{\chi}_i \in \mathcal{U}_{\rm ad}$, $0 \le \sigma_i \le M$, and the Young inequality. Here the constant C depends on M, k_1^*, k_{2*}, R, S^* and the Lipschitz constant of k_1, k_2, S. It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |\nabla\sigma|^2\,\mathrm{d}x + \frac{1}{2}\int_\Gamma |\sigma|^2\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad \le \frac{1}{2}\int_\Gamma |\sigma_\Gamma|^2\,\mathrm{d}\mathcal{H}^{d-1} + C\int_\Omega \left(|\varphi|^2 + |z|^2 + |\chi_2|^2 + |\sigma|^2\right)\,\mathrm{d}x.\end{aligned} \tag{5.1.86}$$

We take the difference of (5.1.23c) written for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and test it by $\partial_t\boldsymbol{u}$. Integrating over Ω, summing and subtracting some terms and using the positive definiteness of $\mathcal{A}$, we get

$$\begin{aligned}C_{\mathcal{A}*}&\int_\Omega |\varepsilon(\partial_t\boldsymbol{u})|^2\,\mathrm{d}x \le \int_\Omega \mathcal{A}(\varphi_1,z_1)\varepsilon(\partial_t\boldsymbol{u}) : \varepsilon(\partial_t\boldsymbol{u})\,\mathrm{d}x\\ &= -\int_\Omega (\mathcal{A}(\varphi_1,z_1) - \mathcal{A}(\varphi_2,z_2))\varepsilon(\partial_t\boldsymbol{u}_2) : \varepsilon(\partial_t\boldsymbol{u})\,\mathrm{d}x\\ &\quad -\int_\Omega \mathcal{B}(\varphi_1,z_1)\varepsilon(\boldsymbol{u}) : \varepsilon(\partial_t\boldsymbol{u})\,\mathrm{d}x\\ &\quad -\int_\Omega (\mathcal{B}(\varphi_1,z_1) - \mathcal{B}(\varphi_2,z_2))\varepsilon(\boldsymbol{u}_2) : \varepsilon(\partial_t\boldsymbol{u})\,\mathrm{d}x + \int_\Omega \boldsymbol{f}\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x\\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{5.1.87}$$

Our next aim is to provide a bound for each integral on the right-hand side. Regarding I_1, we employ the Lipschitz continuity of $\mathcal{A}$ and the Hölder inequality. We obtain

$$\begin{aligned}I_1 &\le C\int_\Omega (|\varphi| + |z|)|\varepsilon(\partial_t\boldsymbol{u}_2)||\varepsilon(\partial_t\boldsymbol{u})|\,\mathrm{d}x\\ &\le C(\|\varphi\|_{L^3(\Omega)} + \|z\|_{L^3(\Omega)})\|\varepsilon(\partial_t\boldsymbol{u}_2)\|_{L^6(\Omega)}\|\varepsilon(\partial_t\boldsymbol{u})\|_H\\ &\le \eta\|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 + C_\eta(\|\varphi\|_{L^3(\Omega)}^2 + \|z\|_{L^3(\Omega)}^2),\end{aligned}$$

where the last inequality holds because $\varepsilon(\partial_t\boldsymbol{u}_2)$ is uniformly bounded in $L^\infty(V) \hookrightarrow L^\infty(L^6(\Omega))$ and we have applied the Young inequality with a small constant η. Notice that C_η depends on $\max_{i=1,2}\left(\|\varepsilon(\partial_t\boldsymbol{u}_i)\|_{L^\infty(V)}\right)$. Employing Lemma 2.6 and then again the Young inequality yields

$$\begin{aligned}I_1 &\le \eta\|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 + C_\eta(\|\varphi\|_H^{1/2}\|\varphi\|_V^{1/2} + \|z\|_H^{1/2}\|z\|_V^{1/2})^2\\ &\le \eta(\|\varepsilon(\partial_t\boldsymbol{u})\|_H^2 + \|\varphi\|_V^2 + \|z\|_V^2) + C_\eta(\|\varphi\|_H^2 + \|z\|_H^2)\\ &\le \eta\left(\int_\Omega |\varepsilon(\partial_t\boldsymbol{u})|^2\,\mathrm{d}x + \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \int_\Omega |\nabla z|^2\,\mathrm{d}x\right)\\ &\quad + C_\eta\left(\int_\Omega |\varphi|^2\,\mathrm{d}x + \int_\Omega |z|^2\,\mathrm{d}x\right).\end{aligned} \tag{5.1.88}$$

Concerning I_2, using the Hölder and the Young inequalities, we get that

$$\begin{aligned} I_2 &\leq C \int_\Omega |\varepsilon(\boldsymbol{u})||\varepsilon(\partial_t \boldsymbol{u})| \,\mathrm{d}x \leq \eta \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x + C_\eta \int_\Omega |\varepsilon(\boldsymbol{u})|^2 \,\mathrm{d}x \\ &\leq \eta \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x + C_\eta \Big[\int_\Omega |\varepsilon(\boldsymbol{u}_0)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x \,\mathrm{d}\tau \Big]. \end{aligned} \tag{5.1.89}$$

The integral I_3 can be treated exactly as I_1 and satisfies the same estimate

$$\begin{aligned} I_3 \leq \eta \Big(&\int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x + \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \int_\Omega |\nabla z|^2 \,\mathrm{d}x \Big) \\ &+ C_\eta \Big(\int_\Omega |\varphi|^2 \,\mathrm{d}x + \int_\Omega |z|^2 \,\mathrm{d}x \Big), \end{aligned} \tag{5.1.90}$$

where C_η depends on $\max_{i=1,2} \big(\|\varepsilon(\boldsymbol{u}_i)\|_{L^\infty(V)} \big)$. Finally, I_4 can be handled by employing the Hölder and the Young inequalities again, in order to obtain

$$I_4 \leq \eta \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x + C_\eta \int_\Omega |\boldsymbol{f}|^2 \,\mathrm{d}x. \tag{5.1.91}$$

Taking advantage of inequalities (5.1.88), (5.1.89), (5.1.90), (5.1.91) in (5.1.87) and redefining η, we obtain:

$$\begin{aligned} &C_{\mathcal{A}*} \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x \\ &\leq \eta \Big(\int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x + \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \int_\Omega |\nabla z|^2 \,\mathrm{d}x \Big) + C_\eta \Big(\int_\Omega |\varphi|^2 \,\mathrm{d}x \\ &\quad + \int_\Omega |z|^2 \,\mathrm{d}x + \int_\Omega |\varepsilon(\boldsymbol{u}_0)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega |\boldsymbol{f}|^2 \,\mathrm{d}x \Big). \end{aligned} \tag{5.1.92}$$

Finally, we take the difference between (5.1.23d) written for z_1 and z_2 and test it by z. Integrating over Ω, exploiting monotonicity of β, Lipschitz continuity of π and F and the Young inequality, we deduce that

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |z|^2 \,\mathrm{d}x + \int_\Omega |\nabla z|^2 \,\mathrm{d}x \leq \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |z|^2 \,\mathrm{d}x + \int_\Omega |\nabla z|^2 \,\mathrm{d}x + \int_\Omega (\xi_1 - \xi_2) z \,\mathrm{d}x \\ &= - \int_\Omega (\pi(z_1) - \pi(z_2)) z \,\mathrm{d}x + \int_\Omega \iota \, z \,\mathrm{d}x \\ &\quad - \int_\Omega [F(\varphi_1, \varepsilon(\boldsymbol{u}_1)) - F(\varphi_2, \varepsilon(\boldsymbol{u}_2))] \, z \,\mathrm{d}x \\ &\leq C \Big(\int_\Omega |z|^2 \,\mathrm{d}x + \int_\Omega |\iota|^2 \,\mathrm{d}x + \int_\Omega |\varphi|^2 \,\mathrm{d}x + \int_\Omega |\varepsilon(\boldsymbol{u})|^2 \,\mathrm{d}x \Big), \end{aligned} \tag{5.1.93}$$

where C depends only on the Lipschitz constants of π and F. At this point, summing inequalities (5.1.85), (5.1.86), (5.1.92), (5.1.93) and moving the terms multiplied by η to the left-hand side, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega |\varphi|^2\,\mathrm{d}x + \int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |z|^2\,\mathrm{d}x\bigg) + \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \int_\Omega |\nabla\sigma|^2\,\mathrm{d}x \\
&\quad + \int_\Omega |\nabla z|^2\,\mathrm{d}x + \int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2\,\mathrm{d}x + \int_\Gamma |\sigma|^2\,\mathrm{d}\mathcal{H}^{d-1} \\
&\leq C\bigg[\int_\Omega |\sigma|^2\,\mathrm{d}x + \int_\Omega |z|^2\,\mathrm{d}x + \int_\Omega |\varphi|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\varepsilon(\partial_t \boldsymbol{u})|^2\,\mathrm{d}x\,\mathrm{d}\tau\bigg] \\
&\quad + C\bigg[\int_\Omega |\varepsilon(\boldsymbol{u}_0)|^2\,\mathrm{d}x + \int_\Gamma |\sigma_\Gamma|^2\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega |\boldsymbol{\chi}|^2\,\mathrm{d}x + \int_\Omega |\boldsymbol{f}|^2\,\mathrm{d}x + \int_\Omega |\iota|^2\,\mathrm{d}x\bigg].
\end{aligned}$$

Integrating in time and then applying the Gronwall inequality, the following estimate follows

$$\begin{aligned}
&\|\varphi\|_{L^\infty(H)\cap L^2(V)} + \|\sigma\|_{L^\infty(H)\cap L^2(V)} + \|z\|_{L^\infty(H)\cap L^2(V)} + \|\boldsymbol{u}\|_{H^1(V_0)} \\
&\quad \leq C\Big(\|\varphi_0\|_H + \|\sigma_0\|_H + \|z_0\|_H + \|\boldsymbol{u}_0\|_{V_0} \\
&\qquad + \|\boldsymbol{\chi}\|_{L^2(H)} + \|\boldsymbol{f}\|_{L^2(H)} + \|\iota\|_{L^2(H)} + \|\sigma_\Gamma\|_{L^2(L^2(\Gamma))}\Big)
\end{aligned}$$

for a constant C that does not depend on the differences φ, σ, $\boldsymbol{u}$ and z but depends on the fixed data of the problem, T and $\max_{i=1,2}(\|\varepsilon(\partial_t \boldsymbol{u}_i)\|_{L^\infty(V)})$. This concludes the proof of Theorem 5.8.

Remark 5.13. From Theorem 5.7 and Theorem 5.8, employing standard interpolation results, we are able to prove the following estimates, which we will need later on. Precisely, for every $\boldsymbol{\chi}, \overline{\boldsymbol{\chi}}$ in $\mathcal{U}_R$, we have:

$$\|\varphi-\overline{\varphi}\|_{L^\infty(L^4(\Omega))}+\|\varepsilon(\boldsymbol{u})-\varepsilon(\overline{\boldsymbol{u}})\|_{L^\infty(L^4(\Omega))}+\|z-\overline{z}\|_{L^\infty(L^4(\Omega))} \leq C_R\|\boldsymbol{\chi}-\overline{\boldsymbol{\chi}}\|_{L^2(Q)}^{\frac{1}{4}}. \tag{5.1.94}$$

Applying the Gagliardo–Nirenberg interpolation inequality from Lemma 2.6, as well as the regularity estimate (5.1.24) and the continous dependence inequality from Theorem 5.8, we have:

$$\begin{aligned}
&\|\varepsilon(\boldsymbol{u}) - \varepsilon(\overline{\boldsymbol{u}})\|_{L^\infty(L^4(\Omega))} \leq C\|\varepsilon(\boldsymbol{u}) - \varepsilon(\overline{\boldsymbol{u}})\|_{L^\infty(H)}^{\frac{1}{4}}\|\varepsilon(\boldsymbol{u}) - \varepsilon(\overline{\boldsymbol{u}})\|_{L^\infty(V)}^{\frac{3}{4}} \\
&\quad \leq C\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{L^\infty(V_0)}^{\frac{1}{4}}\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{L^\infty(W_0)}^{\frac{3}{4}} \leq C_R\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{H^1(V_0)}^{\frac{1}{4}} \leq C_R\|\boldsymbol{\chi} - \overline{\boldsymbol{\chi}}\|_{L^2(Q)}^{\frac{1}{4}}.
\end{aligned}$$

The same can be performed for φ and z. Similarly, we have

$$\|\varphi - \overline{\varphi}\|_{L^4(L^\infty(\Omega))} + \|z - \overline{z}\|_{L^4(L^\infty(\Omega))} \leq C_R\|\boldsymbol{\chi} - \overline{\boldsymbol{\chi}}\|_{L^2(Q)}^{\frac{1}{4}}. \tag{5.1.95}$$

In fact, applying Gagliardo–Nirenberg inequality and the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain

$$\|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \leq \|\varphi - \overline{\varphi}\|_{W^{1,4}(\Omega)} \leq C\|\varphi - \overline{\varphi}\|_V^{\frac{1}{4}}\|\varphi - \overline{\varphi}\|_W^{\frac{3}{4}} \leq C_R\|\varphi - \overline{\varphi}\|_V^{\frac{1}{4}},$$

where the last inequality follows from the regularity estimate (5.1.24). Elevating to the power 4th and integrating in time over $(0,T)$ leads to

$$\begin{aligned}\|\varphi-\overline{\varphi}\|^4_{L^4(L^\infty(\Omega))} &= \int_0^T \|\varphi-\overline{\varphi}\|^4_{L^\infty(\Omega)}\,\mathrm{d}t \le C_R \int_0^T \|\varphi-\overline{\varphi}\|_V\,\mathrm{d}t \\ &= C_R\|\varphi-\overline{\varphi}\|_{L^1(V)} \le C_R\|\varphi-\overline{\varphi}\|_{L^2(V)} \le C_R\|\boldsymbol{\chi}-\overline{\boldsymbol{\chi}}\|_{L^2(Q)}\end{aligned}$$

thanks to the continuous dependence estimate from Theorem 5.8. The same holds for the damage variable. Finally, the following inequality is satisfied

$$\|\varphi-\overline{\varphi}\|_{L^4(L^3(\Omega))} + \|z-\overline{z}\|_{L^4(L^3(\Omega))} \le C_R\|\boldsymbol{\chi}-\overline{\boldsymbol{\chi}}\|_{L^2(Q)}. \tag{5.1.96}$$

Proceeding as before, we gather

$$\begin{aligned}\|\varphi-\overline{\varphi}\|^4_{L^4(L^3(\Omega))} &= \int_0^T \|\varphi-\overline{\varphi}\|^4_{L^3(\Omega)}\,\mathrm{d}t \le C\int_0^T \|\varphi-\overline{\varphi}\|^2_H\|\varphi-\overline{\varphi}\|^2_V\,\mathrm{d}t \\ &\le C\|\varphi-\overline{\varphi}\|^2_{L^\infty(H)}\int_0^T \|\varphi-\overline{\varphi}\|^2_V\,\mathrm{d}t \\ &= C\|\varphi-\overline{\varphi}\|^2_{L^\infty(H)}\|\varphi-\overline{\varphi}\|^2_{L^2(V)} \le C_R\|\boldsymbol{\chi}-\overline{\boldsymbol{\chi}}\|^4_{L^2(Q)},\end{aligned}$$

and the same for z.

5.2 The optimal control problem

Assume that hypotheses (A1)–(A9) and (B1)–(B4) hold. In view of the well-posedness of the state system proved in Theorems 5.5 and 5.8, we introduce the so-called *control-to-state operator* or *solution operator*, which maps every control $\boldsymbol{\chi}$ to the corresponding unique state and is denoted by $\mathcal{S}$. More precisely, we introduce the state-space

$$\begin{aligned}\mathcal{V}_{\mathcal{S}} \coloneqq &\left[H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\right] \\ &\times\left[H^1(0,T;V')\cap L^\infty(0,T;H)\cap L^2(0,T;V)\right]\times W^{1,\infty}(0,T;W) \\ &\times\left[H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\right]\end{aligned}$$

which is continuously embedded into the larger

$$\begin{aligned}\mathcal{V} \coloneqq &\left[C^0([0,T];H)\cap L^2(0,T;V)\right]\times\left[C^0([0,T];H)\cap L^2(0,T;V)\right] \\ &\times H^1(0,T;V_0)\times\left[C^0([0,T];H)\cap L^2(0,T;V)\right].\end{aligned}$$

We define

$$\mathcal{S}:\mathcal{U}\to\mathcal{V},\qquad \boldsymbol{\chi}\mapsto(\varphi,\sigma,\boldsymbol{u},z),$$

where $(\varphi,\sigma,\boldsymbol{u},z)$ is the unique solution of the state system (1.3.14)–(1.3.16) obtained choosing $\boldsymbol{\chi}$ as the control. Notice that $\mathcal{S}$ is well-defined over $\mathcal{U}$ and Lipschitz continuous over $\mathcal{U}_R$ because of Theorem 5.8. We introduce the reduced cost functional as

$$\mathrm{J}(\boldsymbol{\chi}) \coloneqq \mathcal{J}(\mathcal{S}(\boldsymbol{\chi}),\boldsymbol{\chi}).$$

Then, the optimal control problem can be stated as

$$\min_{\boldsymbol{\chi}\in\mathcal{U}_{\mathrm{ad}}} \mathrm{J}(\boldsymbol{\chi}), \tag{5.2.1}$$

which means that we search a minimizer for the functional $\mathcal{J}$ subject to the PDE system (1.3.14)–(1.3.16) and constrained to $\mathcal{U}_{\mathrm{ad}}$.

In order to prove that (5.2.1) admits a minimizer and to derive the associated first-order necessary conditions, the assumptions (A1)–(A9) and (B1)–(B4) alone are not sufficient. To keep the formulation of the hypotheses clear and easily accessible, we retain the framework given by assumptions (A1)–(A9), while replacing (B1)–(B4) entirely with the new set presented below. These conditions include the regularity requirements on the assigned functions and initial data necessary for the application of Theorems 5.7 and 5.8, as well as further assumptions tailored to the control problem. In particular, the enhanced regularity of the nonlinearities is essential for handling the corresponding terms in the linearized and adjoint systems.

(C1) The nonlinear terms from the tumor and the lactate equations have the following regularity:

$$p,\, g \in W^{1,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2), \tag{5.2.2}$$

$$k_1, k_2, S \in W^{1,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2). \tag{5.2.3}$$

(C2) The viscous tensor $\mathcal{A} = (a_{ijkh})$ is constant, while the elastic tensor $\mathcal{B} = (b_{ijkh}) : \mathbb{R}^2 \to \mathbb{R}^{n\times n\times n\times n}$ satisfy:

$$\mathcal{B} \in W^{1,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2). \tag{5.2.4}$$

Moreover, being able to bound the maximal monotone operator β and its higher derivatives will be essential. To this end, the key point is proving a strict separation property for the damage variable z. This leads us to adopt a more restrictive form of the convex potential $\hat{\beta}$, moving beyond the quite general hypotheses employed in the previous analysis.

(C3) We assume that the convex part of the potential has the regularity

$$\widehat{\beta} \in C^3(0,1), \tag{5.2.5}$$

and that its derivative β satisfies the growth conditions:

$$\lim_{r\to 0^+} \beta(r) = -\infty, \qquad \lim_{r\to 1^-} \beta(r) = +\infty. \tag{5.2.6}$$

Regarding the nonconvex part of the potential, we require

$$\widehat{\pi} \in C^{1,1}(\mathbb{R}) \cap C^3(0,1). \tag{5.2.7}$$

Example 5.14. The prototypical example to keep in mind is the logarithmic potential

$$\hat{\beta}(r) + \hat{\pi}(r) = C_1 \left[r \ln r + (1-r) \ln (1-r) \right] + C_2 r(1-r)$$

for some given and positive constants C_1, C_2, which clearly fulfills hypothesis (C3).

To establish the separation property for the damage variable, we will also need to impose that the functions ι and $F(\varphi, \varepsilon(\boldsymbol{u}))$ are bounded.

(C4) We suppose that $\iota \in L^\infty(Q) \cap H^1(0,T;H)$.

(C5) We assume that $F \in W^{2,\infty}(\Omega \times \mathbb{R} \times \mathbb{R}^{n\times n})$.

Remark 5.15. It is worth noting that this assumption is not as restrictive as it might initially appear. Indeed, we will prove the boundedness of φ, and since we are working within the framework of linear elasticity, $\varepsilon(\boldsymbol{u})$ is expected to remain small. Consequently, under hypothesis (5.1.15), the boundedness of $F(\varphi, \varepsilon(\boldsymbol{u}))$ would follow asking that $\hat{F} = F(0, \mathbf{0}) \in L^\infty(\Omega)$, instead of $\hat{F} \in H$, as we did in (C5). However, as we are unable to prove the boundedness of $\varepsilon(\boldsymbol{u})$ mathematically, it becomes necessary to explicitly impose the boundedness of F as an additional condition. The boundedness of the higher derivatives is required to handle the linearized coefficients in the linearized and adjoint systems.

Finally, since the domain of $\hat{\beta}$ is $[0,1]$, the initial datum z_0 must be chosen such that its values lie within this interval. Recalling again that our goal is to prove a separation property for the damage variable, we request that z_0 stays bounded away from 0 and 1.

(C6) We assume that

$$\varphi_0 \in W, \quad \boldsymbol{u}_0 \in W_0, \quad z_0 \in W, \quad 0 < \operatorname*{ess\,inf}_{\Omega}(z_0), \quad \operatorname*{ess\,sup}_{\Omega}(z_0) < 1. \tag{5.2.8}$$

Regarding the cost functional $\mathcal{J}$, we make the following assumptions.

(C7) The coefficients $\alpha_1, \ldots, \alpha_9$ are nonnegative constants that cannot vanish all at the same time.

(C8) The target functions satisfy

$$\varphi_Q, \sigma_Q, z_Q \in L^2(Q), \qquad \varphi_\Omega, \sigma_\Omega \in H. \tag{5.2.9}$$

(C9) The coefficient $\gamma : \Omega \times \mathbb{R} \to [0, +\infty)$ is a Carathéodory function such that

$$\gamma(x, \cdot) \in C^1(\mathbb{R}) \tag{5.2.10}$$

for a.e. $x \in \Omega$. Moreover, it exists a constant $C_\gamma > 0$ such that

$$|\gamma(x,\varphi)| + |\partial_\varphi \gamma(x,\varphi)| \le C_\gamma \tag{5.2.11}$$

for a.e. $x \in \Omega$ and for all $\varphi \in [0, N]$.

Even if γ also depends on the point $x \in \Omega$, in the following, we will employ the shorter notation $\gamma(\varphi)$ instead of $\gamma(x,\varphi)$. For the same reason, the partial derivative of γ with respect to the variable φ will be denoted by $\gamma'(\varphi)$.

5.2.1 A strict separation property for the damage

As already mentioned when introducing the additional assumptions, the more restrictive hypotheses on the potential $\hat{\beta}$, as well as boundedness for ι and F, enable us to establish a separation property for the damage z.

Proposition 5.16. *Let assumptions (A1)–(A9) and (C1)–(C9) hold. Then, there exist $0 < r_* \le r^* < 1$ which may depend on the data of the problem and on R such that, for every $\boldsymbol{\chi} \in \mathcal{U}_R$ with the associated solution to the state problem $(\varphi, \sigma, \boldsymbol{u}, z)$, there holds*

$$r_* \le z \le r^*$$

a.e. in Q.

Proof. We claim that there exist $0 < r_* \le r^* < 1$ such that

(i) $r_* \le \operatorname*{ess\,inf}_\Omega (z_0)$,

(ii) $\operatorname*{ess\,sup}_\Omega (z_0) \le r^*$,

(iii) $\beta(r) + \pi(r) + \|\iota\|_{L^\infty(Q)} + \|F\|_{L^\infty(Q)} \le 0$ for all $r \in (0, r_*)$,

(iv) $\beta(r) + \pi(r) - \|\iota\|_{L^\infty(Q)} - \|F\|_{L^\infty(Q)} \ge 0$ for all $r \in (r^*, 1)$.

We can find such r_*, r^* satisfying conditions (i) and (ii) because of hypothesis (C6), ensuring that z_0 remains bounded away from 0 and 1. Regarding points (iii) and (iv), we recall that, from the specific choice we made for the potential $\hat{\beta}$, it holds

$$\beta(r) \to -\infty \text{ if } r \to 0^+, \qquad \beta(r) \to +\infty \text{ if } r \to 1^-.$$

Moreover, π is bounded over the interval $[0, 1]$ because $\pi \in C^0(\mathbb{R})$ from hypothesis (C3), and ι and F are bounded respectively from hypotheses (C4) and (C5). Let $\boldsymbol{\chi}$ be an arbitrary control in $\mathcal{U}_R$ and $\mathcal{S}(\boldsymbol{\chi}) = (\varphi, \sigma, \boldsymbol{u}, z)$. We test equation (5.1.23d) with $(z - r^*)^+$, obtaining:

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |(z - r^*)^+|^2 \,\mathrm{d}x + \int_\Omega |\nabla (z - r^*)^+|^2 \,\mathrm{d}x \\ &\quad + \int_\Omega [\beta(z) + \pi(z) - \iota + F(\varphi, \varepsilon(\boldsymbol{u}))]\,(z - r^*)^+ \,\mathrm{d}x \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |(z - r^*)^+|^2 \,\mathrm{d}x, \end{aligned}$$

where we used property (iv) in order to obtain the inequality. Integrating in time over the interval $(0, t)$, it follows that

$$\int_\Omega |(z - r^*)^+|^2 \,\mathrm{d}x \le \int_\Omega |(z_0 - r^*)^+|^2 \,\mathrm{d}x = 0$$

because, according to property (ii), z_0 is smaller or equal to r^* almost everywhere, thus $(z_0 - r^*)^+ = 0$. This proves that $(z - r^*)^+ = 0$ and, equivalently, that z is smaller than or equal to r^* almost everywhere in Q. Proceeding in the same way, we test equation (5.1.23d) with $-(z - r_*)^-$. Employing inequality (iii) and then integrating in time and using (i), we obtain that $(z - r_*)^-$ is equal to 0. Thus, we have that z is greater than or equal to r_* almost everywhere in Q. ∎

Remark 5.17. Let us highlight the fact that, thanks to the strict separation property we have just proved, from now on we will treat β as a regular potential. Moreover, we trivially deduce that

$$\|\beta(z)\|_{L^\infty(Q)} + \|\beta'(z)\|_{L^\infty(Q)} + \|\beta''(z)\|_{L^\infty(Q)} \leq C_R,$$

since $\beta \in C^2(0,1)$ by hypothesis (C3).

5.2.2 Existence of a minimizer

We are now in the position to prove the existence of at least one solution to the optimal control problem (5.2.1). The proof is an application of the direct method of the Calculus of Variations.

Theorem 5.18. *Suppose that hypotheses (A1)–(A9) and (C1)–(C9) hold. Then, there exists at least one minimizer $\boldsymbol{\chi}^* \in \mathcal{U}_{ad}$ to the optimal control problem (5.2.1).*

Proof. The reduced cost functional is proper and non-negative, so $\inf_{\boldsymbol{\chi}\in\mathcal{U}_{\text{ad}}} \mathrm{J}(\boldsymbol{\chi})$ is finite and non-negative. Let $\{\boldsymbol{\chi}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}_{\text{ad}}$ be a minimizing sequence for J, meaning that

$$\inf_{\boldsymbol{\chi}\in\mathcal{U}_{\text{ad}}} \mathrm{J}(\boldsymbol{\chi}) = \lim_{n\to+\infty} \mathrm{J}(\boldsymbol{\chi}_n).$$

We denote the corresponding solution to the state system as $(\varphi_n, \sigma_n, \boldsymbol{u}_n, z_n) \coloneqq \mathcal{S}(\boldsymbol{\chi}_n)$. Since the sequence $\{\boldsymbol{\chi}_n\}_{n\in\mathbb{N}} \in \mathcal{U}_{\text{ad}}$, it is uniformly bounded in $\mathcal{U}$ and, consequently, there exists a $\boldsymbol{\chi}^* \in \mathcal{U}$ such that, along a subsequence that we do not relabel,

$$\chi_{n,1} \to \chi_1^* \quad \text{weakly-}* \quad \text{in } L^2(0,T;V) \cap L^\infty(Q), \tag{5.2.12}$$
$$\chi_{n,2} \to \chi_2^* \quad \text{weakly-}* \quad \text{in } L^\infty(Q). \tag{5.2.13}$$

Notice that $\mathcal{U}_{\text{ad}}$ is convex and closed in the space $L^2(0,T;V) \times L^2(Q)$; thus, it is sequentially weakly closed. This justifies the fact that the limit $\boldsymbol{\chi}^*$ belongs to $\mathcal{U}_{\text{ad}}$. Moreover, by the uniform boundedness of the solution sequence from Theorem 5.7, applying Banach–Alaouglu (see, e.g., [Bré11]) and Aubin–Lions Theorems (see Lemma 2.8), there exists a quadruple $(\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*) \in \mathcal{V}_{\mathcal{S}}$ such that, along a further subsequence,

$$\varphi_n \to \varphi^* \quad \text{weakly-}* \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.2.14}$$
$$\text{strong} \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{5.2.15}$$
$$\text{a.e.} \quad \text{in } Q, \tag{5.2.16}$$
$$\sigma_n \to \sigma^* \quad \text{weakly-}* \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.2.17}$$
$$\text{strong} \quad \text{in } C^0([0,T];V') \cap L^2(0,T;H), \tag{5.2.18}$$
$$\text{a.e.} \quad \text{in } Q, \tag{5.2.19}$$
$$\boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{weakly-}* \quad \text{in } W^{1,\infty}(0,T;W_0), \tag{5.2.20}$$
$$\text{strong} \quad \text{in } C^0([0,T];V_0), \tag{5.2.21}$$

$$
\begin{aligned}
z_n \to z^* \qquad & \text{weakly-}* && \text{in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), && (5.2.22)\\
& \text{strong} && \text{in } C^0([0,T];H) \cap L^2(0,T;V), && (5.2.23)\\
& \text{a.e.} && \text{in } Q. && (5.2.24)
\end{aligned}
$$

The first step of the proof consists in proving that $\mathcal{S}(\boldsymbol{\chi}^*) = (\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*)$. To do so, thanks to the convergences above, we can pass to the limit in the PDE system satisfied by $(\varphi_n, \sigma_n, \boldsymbol{u}_n, z_n)$ and $\boldsymbol{\chi}_n$, whence $(\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*)$ is the unique solution to the state system associated with $\boldsymbol{\chi}^*$. More precisely, notice that the passage to the limit in the linear terms is trivial because of the weak convergences (5.2.14), (5.2.17), (5.2.20), and (5.2.22). Regarding the nonlinear ones, we recall that, according to our assumptions, the assigned functions $p, \chi_1, g, k_1, k_2, \chi_2, S, \mathcal{B}, F$ are continuous and bounded, and $\beta \in C^0(0,1)$, and $\pi \in C^0(\mathbb{R})$ at least. Moreover, we proved that φ_n, σ_n, and z_n are uniformly bounded, namely

$$
0 \le \varphi_n \le N, \quad 0 \le \sigma_n \le M, \quad 0 < r_* \le z_n \le r^* < 1
$$

a.e. in Q. Thus, the pointwise convergences (5.2.16), (5.2.19), (5.2.24), and the Dominated Convergence Theorem are enough to pass to the limit in the nonlinear terms.
The second step is showing that

$$
\inf_{\boldsymbol{\chi} \in \mathcal{U}_{\mathrm{ad}}} \mathrm{J}(\boldsymbol{\chi}) = \mathrm{J}(\boldsymbol{\chi}^*).
$$

To this end, we write the reduced cost functional as the sum of the following terms

$$
\begin{aligned}
\mathrm{J}_1(\boldsymbol{\chi}) &= \frac{\alpha_1}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)} + \frac{\alpha_4}{2}\|\sigma - \sigma_Q\|^2_{L^2(Q)} + \frac{\alpha_7}{2}\|z - z_Q\|^2_{L^2(Q)} + \frac{\alpha_9}{2}\|\boldsymbol{\chi}\|^2_{L^2(Q)},\\
\mathrm{J}_2(\boldsymbol{\chi}) &= \frac{\alpha_2}{2}\|\varphi(T) - \varphi_\Omega\|^2_H + \alpha_3\|\varphi(T)\|_{L^1(\Omega)}\\
&\quad + \frac{\alpha_5}{2}\|\sigma(T) - \sigma_\Omega\|^2_H + \frac{\alpha_6}{2}\|\sqrt{\gamma(\varphi)}\varepsilon(\boldsymbol{u})\|^2_{L^2(Q)} + \alpha_8\|z(T)\|_{L^1(\Omega)}.
\end{aligned}
$$

By the weak convergences (5.2.12)–(5.2.14), (5.2.17), (5.2.22), and weak lower semicontinuity of the L^2-norm, we obtain

$$
\liminf_{n\to+\infty} \mathrm{J}_1(\boldsymbol{\chi}_n) \ge \mathrm{J}_1(\boldsymbol{\chi}^*).
$$

Again from Theorem 5.7, $\{\sigma^*(T)\}_{n\in\mathbb{N}}$ is uniformly bounded in H. Thus, extracting a further subsequence,

$$
\sigma_n(T) \to \sigma^*(T) \qquad \text{weakly} \qquad \text{in } H, \tag{5.2.25}
$$

where we are able to identify the limit with $\sigma^*(T)$ because of convergence (5.2.18). Since $\varphi_n \to \varphi^*$ a.e. by (5.2.16), and γ is bounded by hypothesis (C9),

$$
\sqrt{\gamma(\varphi_n)}\eta \to \sqrt{\gamma(\varphi^*)}\eta \qquad \text{strongly} \qquad \text{in } L^2(0,T;H), \tag{5.2.26}
$$

for every $\eta \in L^2(0,T;H)$ thanks to the Dominated Convergence Theorem. Moreover, from the convergence (5.2.21), we know that

$$
\varepsilon(\boldsymbol{u}_n) \to \varepsilon(\boldsymbol{u}^*) \qquad \text{strongly} \qquad \text{in } L^2(0,T;H). \tag{5.2.27}
$$

Combining (5.2.26) and (5.2.27), we deduce that

$$\sqrt{\gamma(\varphi_n)}\varepsilon(\boldsymbol{u}_n) \to \sqrt{\gamma(\varphi^*)}\varepsilon(\boldsymbol{u}^*) \qquad \text{weakly} \qquad \text{in } L^2(0,T;H). \tag{5.2.28}$$

By the weak lower semicontinuity of the L^2-norm and the weak convergences (5.2.25), (5.2.28), and by the strong continuity of the L^2-norm and the strong convergences (5.2.15), (5.2.23), we have

$$\liminf_{n\to+\infty} \mathrm{J}_2(\boldsymbol{\chi}_n) \geq \mathrm{J}_2(\boldsymbol{\chi}^*)$$

whence the thesis. □

We aim to establish first-order optimality conditions for the optimal control problem (5.2.1). To do so, the standard procedure is to prove the Fréchet differentiability of the control-to-state operator. With this purpose, we linearize the state system.

5.3 The linearized state system

We consider a fixed control $\boldsymbol{\chi} \in \mathcal{U}_{\mathrm{ad}}$ with the associated state given by $(\varphi, \sigma, \boldsymbol{u}, z) = \mathcal{S}(\boldsymbol{\chi})$. For every small $\boldsymbol{h} = (h_1, h_2) \in \mathcal{U}$, we consider the perturbed variables

$$\varphi + \xi, \quad \sigma + \rho, \quad \boldsymbol{u} + \boldsymbol{\omega}, \quad z + \zeta, \quad \boldsymbol{\chi} + \boldsymbol{h},$$

and the corresponding state system. Linearizing it near $(\varphi, \sigma, \boldsymbol{u}, z)$, $\boldsymbol{\chi}$ and approximating the nonlinearities with their first order Taylor expansions, we have that $(\xi, \rho, \boldsymbol{\omega}, \zeta)$, $\boldsymbol{h}$ satisfy the linear PDE system

$$\partial_t \xi - \Delta \xi = a_1 \xi + a_2 \rho + a_3 \zeta + a_4 h_1, \tag{5.3.1a}$$
$$\partial_t \rho - \Delta \rho = b_1 \xi + b_2 \rho + b_3 \zeta + b_4 h_2, \tag{5.3.1b}$$
$$-\operatorname{div}\left[\mathcal{A}\varepsilon(\partial_t \boldsymbol{\omega}) + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{\omega})\right] = -\operatorname{div}\left[\mathfrak{c}_1 \xi + \mathfrak{c}_2 \zeta\right], \tag{5.3.1c}$$
$$\partial_t \zeta - \Delta \zeta = d_1 \xi + \mathfrak{d}_2 : \varepsilon(\boldsymbol{\omega}) + d_3 \zeta, \tag{5.3.1d}$$

where, for the sake of better readability, we introduced the following notation:

$$\begin{aligned}
a_1 &= U_{,\varphi} = (p(\sigma,z) - \chi_1)\left(1 - 2\frac{\varphi}{N}\right) - g(\sigma,z),\\
a_2 &= U_{,\sigma} = p_{,\sigma}(\sigma,z)\varphi\left(1 - \frac{\varphi}{N}\right) - \varphi g_{,\sigma}(\sigma,z),\\
a_3 &= U_{,z} = p_{,z}(\sigma,z)\varphi\left(1 - \frac{\varphi}{N}\right) - \varphi g_{,z}(\sigma,z),\\
a_4 &= U_{,\chi_1} = -\varphi\left(1 - \frac{\varphi}{N}\right),\\
b_1 &= -K_{,\varphi}(\varphi,\sigma,z) + \chi_2 S_{,\varphi}(\varphi,z)\\
&= -\frac{k_{1,\varphi}(\varphi,z)\sigma}{k_2(\varphi,z)+\sigma} + \frac{k_1(\varphi,z)\sigma k_{2,\varphi}(\varphi,z)}{(k_2(\varphi,z)+\sigma)^2} + \chi_2 S_{,\varphi}(\varphi,z),\\
b_2 &= -K_{,\sigma}(\varphi,\sigma,z) = -\frac{k_1(\varphi,z)}{k_2(\varphi,z)+\sigma} + \frac{k_1(\varphi,z)\sigma}{(k_2(\varphi,z)+\sigma)^2},
\end{aligned}$$

$$
\begin{aligned}
b_3 &= -K_{,z}(\varphi,\sigma,z) + \chi_2 S_{,z}(\varphi,z) \\
&= -\frac{k_{1,z}(\varphi,z)\sigma}{k_2(\varphi,z)+\sigma} + \frac{k_1(\varphi,z)\sigma k_{2,z}(\varphi,z)}{(k_2(\varphi,z)+\sigma)^2} + \chi_2 S_{,z}(\varphi,z), \\
b_4 &= S(\varphi,z), \\
\mathfrak{c}_1 &= -\mathcal{B}_{,\varphi}(\varphi,z)\varepsilon(\boldsymbol{u}), \\
\mathfrak{c}_2 &= -\mathcal{B}_{,z}(\varphi,z)\varepsilon(\boldsymbol{u}), \\
d_1 &= -F_{,\varphi}(\varphi,\varepsilon(\boldsymbol{u})), \\
\mathfrak{d}_2 &= -F_{,\varepsilon}(\varphi,\varepsilon(\boldsymbol{u})), \\
d_3 &= -\beta'(z) - \pi'(z).
\end{aligned}
$$

Remark 5.19. We first note that the terms written in Fraktur font are matrix-valued functions, whereas the remaining ones are scalar-valued. Second, we make some remarks on their regularity. Since all the assigned functions are Lipschitz continuous and bounded, and because of the regularity we have already proved for $(\varphi,\sigma,\boldsymbol{u},z)$ in Theorem 5.7, we have

$$
a_1,\ldots,a_4,b_1,\ldots,b_4,d_1,\mathfrak{d}_2 \in L^\infty(Q),
$$

and they are uniformly bounded by a constant that depends on R. Moreover,

$$
\mathfrak{c}_1,\mathfrak{c}_2 \in L^\infty(0,T;L^p(\Omega))
$$

for any $p\in[1,6]$ and their norm is uniformly bounded by a certain C_R, because

$$
|\mathfrak{c}_1| + |\mathfrak{c}_2| \le C|\varepsilon(\boldsymbol{u})|,
$$

and $\boldsymbol{u}\in W^{1,\infty}(0,T;W)\hookrightarrow W^{1,\infty}(0,T;W^{1,p}(\Omega))$. Regarding the last term, thanks to the regularity of β in its domain and to the separation property we proved in Proposition 5.16,

$$
d_3 \in L^\infty(Q),
$$

and its norm is bounded by a constant that depends on R.

We couple system (5.3.1) with the following boundary and initial conditions

$$
\partial_{\boldsymbol{\nu}}\xi = \partial_{\boldsymbol{\nu}}\zeta = 0 \qquad \partial_{\boldsymbol{\nu}}\rho = -\rho, \qquad \boldsymbol{\omega} = 0 \qquad \text{on } \Sigma, \tag{5.3.2}
$$

$$
\xi(0) = \rho(0) = \zeta(0) = 0, \qquad \boldsymbol{\omega}(0) = \boldsymbol{0} \qquad \text{in } \Omega. \tag{5.3.3}
$$

Notice that, even though for the formal derivation of the linearized system we started from a small perturbation $\boldsymbol{h}\in\mathcal{U}_{\mathrm{ad}}$, the obtained system (5.3.1)–(5.3.3) makes sense for every $\boldsymbol{h}\in L^2(Q)\times L^2(Q)$.

Proposition 5.20. *For every $\boldsymbol{\chi}\in\mathcal{U}_R$ with associated state $\mathcal{S}(\boldsymbol{\chi}) = (\varphi,\sigma,\boldsymbol{u},z)\in\mathcal{V}$ and for every $\boldsymbol{h}\in L^2(Q)\times L^2(Q)$ there exists a unique solution $(\xi,\rho,\boldsymbol{\omega},\zeta)$ to the linearized state system* (5.3.1)–(5.3.3) *in the sense that*

$$
\xi \in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W),
$$

$$\rho \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V),$$
$$\boldsymbol{\omega} \in W^{1,\infty}(0,T;V_0),$$
$$\zeta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$

with

$$\xi(0) = \rho(0) = \zeta(0) = 0, \qquad \boldsymbol{\omega}(0) = \mathbf{0}$$

such that

$$\int_\Omega \partial_t \xi \eta \,\mathrm{d}x + \int_\Omega \nabla\xi \cdot \nabla\eta \,\mathrm{d}x = \int_\Omega [a_1\xi + a_2\rho + a_3\zeta + a_4 h_1]\eta \,\mathrm{d}x, \tag{5.3.4a}$$

$$\langle \partial_t \rho, \eta\rangle_V + \int_\Omega \nabla\rho \cdot \nabla\eta \,\mathrm{d}x + \int_\Gamma \rho\eta \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega [b_1\xi + b_2\rho + b_3\zeta + b_4 h_2]\eta \,\mathrm{d}x, \tag{5.3.4b}$$

$$\int_\Omega [\mathcal{A}\varepsilon(\partial_t\boldsymbol{\omega}) + \mathcal{B}(\varphi,z)\varepsilon(\boldsymbol{\omega})] : \varepsilon(\boldsymbol{\theta}) \,\mathrm{d}x = \int_\Omega [\mathfrak{c}_1\xi + \mathfrak{c}_2\zeta] : \varepsilon(\boldsymbol{\theta}) \,\mathrm{d}x, \tag{5.3.4c}$$

$$\int_\Omega \partial_t\zeta\eta \,\mathrm{d}x + \int_\Omega \nabla\zeta \cdot \nabla\eta \,\mathrm{d}x = \int_\Omega [d_1\xi + \mathfrak{d}_2 : \varepsilon(\boldsymbol{\omega}) + d_3\zeta]\,\eta \,\mathrm{d}x, \tag{5.3.4d}$$

a.e. in $(0,T)$*, for every* $\eta \in V$ *and* $\boldsymbol{\theta} \in V_0$*. Moreover, the solution satisfies the following estimate:*

$$\begin{aligned} &\|\xi\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} + \|\rho\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \\ &\quad + \|\boldsymbol{\omega}\|_{W^{1,\infty}(V_0)} + \|\xi\|_{H^1(H)\cap L^\infty(V)\cap L^2(W)} \leq C_R\|\boldsymbol{h}\|_{L^2(Q)}. \end{aligned} \tag{5.3.5}$$

Proof. Existence can be proved using a Galerkin scheme. Since it is a standard procedure, we will show only the formal a priori estimates that are necessary to pass to the limit from the discrete to the continuous system. We test (5.3.1a) with ξ, (5.3.1b) with ρ, (5.3.1d) with ζ, and sum the three equations. Applying the Young inequality and Remark 5.19, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\xi\|_H^2 + \|\rho\|_H^2 + \|\zeta\|_H^2\right) + \|\nabla\xi\|_H^2 + \|\nabla\rho\|_H^2 + \|\nabla\zeta\|_H^2 \\ &\quad \leq C_R\left(\|\xi\|_H^2 + \|\rho\|_H^2 + \|\varepsilon(\boldsymbol{\omega})\|_H^2 + \|\zeta\|_H^2 + \|h_1\|_H^2 + \|h_2\|_H^2\right). \end{aligned} \tag{5.3.6}$$

Testing (5.3.1c) with $\partial_t\boldsymbol{\omega}$ and employing the fact that $\mathcal{A}$ is positive definite, we have

$$\begin{aligned} C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 &\leq \int_\Omega \mathcal{A}\varepsilon(\partial_t\boldsymbol{\omega}) : \varepsilon(\partial_t\boldsymbol{\omega}) \,\mathrm{d}x \\ &= \int_\Omega \left(-B(\varphi,z)\varepsilon(\boldsymbol{\omega}) : \varepsilon(\partial_t\boldsymbol{\omega}) + \xi\mathfrak{c}_1 : \varepsilon(\partial_t\boldsymbol{\omega}) + \zeta\mathfrak{c}_2 : \varepsilon(\partial_t\boldsymbol{\omega})\right) \mathrm{d}x. \end{aligned}$$

Recalling that $\mathcal{B}$ is bounded and $\mathfrak{c}_1, \mathfrak{c}_2$ are uniformly bounded in $L^\infty(0,T;L^6(\Omega))$ thanks to Remark 5.19, we estimate the right-hand side with the Hölder and the Young inequalities. We get

$$\begin{aligned} C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 &\leq \left(C\|\varepsilon(\boldsymbol{\omega})\|_H + \|\xi\|_{L^3(\Omega)}\|\mathfrak{c}_1\|_{L^6(\Omega)} + \|\zeta\|_{L^3(\Omega)}\|\mathfrak{c}_2\|_{L^6(\Omega)}\right)\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H \\ &\leq \delta\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 + C_{R,\delta}\left(\|\varepsilon(\boldsymbol{\omega})\|_H^2 + \|\xi\|_{L^3(\Omega)}^2 + \|\zeta\|_{L^3(\Omega)}^2\right), \end{aligned}$$

where δ is a small positive constant yet to be determined. By means of the interpolation inequality in Lemma 2.6 and again the Young inequality, we have

$$\begin{aligned} C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 \leq \delta\left(\|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 + \|\nabla\xi\|_H^2 + \|\nabla\zeta\|_H^2\right) \\ + C_{R,\delta}\left(\|\varepsilon(\boldsymbol{\omega})\|_H^2 + \|\xi\|_H^2 + \|\zeta\|_H^2\right). \end{aligned} \tag{5.3.7}$$

Recalling that $\boldsymbol{\omega}(0) = 0$,

$$\varepsilon(\boldsymbol{\omega}) = \varepsilon(\boldsymbol{\omega}(0)) + \int_0^t \varepsilon(\partial_t\boldsymbol{\omega})\,\mathrm{d}s = \int_0^t \varepsilon(\partial_t\boldsymbol{\omega})\,\mathrm{d}s.$$

Thus, it follows

$$\|\varepsilon(\boldsymbol{\omega})\|_H^2 \leq C\int_0^t \|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2\,\mathrm{d}s.$$

Summing inequalities (5.3.6) and (5.3.7), choosing δ sufficiently small, and estimating the term $\|\varepsilon(\boldsymbol{\omega})\|_H^2$ on the right-hand side as just shown, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\xi\|_H^2 + \|\rho\|_H^2 + \|\zeta\|_H^2\right) + \|\nabla\xi\|_H^2 + \|\nabla\rho\|_H^2 + \|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2 + \|\nabla\zeta\|_H^2 \\ &\quad \leq C_R\left(\|\xi\|_H^2 + \|\rho\|_H^2 + \|\zeta\|_H^2 + \int_0^t \|\varepsilon(\partial_t\boldsymbol{\omega})\|_H^2\,\mathrm{d}s + \|\boldsymbol{h}\|_{L^2(Q)}^2\right). \end{aligned}$$

Integrating in time over $(0,t)$ and applying Gronwall's inequality, we infer that

$$\|\xi\|_{L^\infty(H)\cap L^2(V)} + \|\rho\|_{L^\infty(H)\cap L^2(V)} + \|\zeta\|_{L^\infty(H)\cap L^2(V)} + \|\boldsymbol{\omega}\|_{H^1(V_0)} \leq C_R\|\boldsymbol{h}\|_{L^2(Q)}.$$

Then, we recover the higher order estimates claimed for ξ, ρ, and ζ by standard parabolic regularity results (see, e.g., [DL92]). Finally, we look back at equation (5.3.7), obtaining the desired uniform bound for $\boldsymbol{\omega}$. These a priori estimates are enough to pass in the Galerkin discretization. This way, existence is proved as well as estimate (5.3.5) in the statement. Uniqueness follows from the energy inequality (5.3.5) and the fact that the system is linear. □

Remark 5.21. For $\boldsymbol{\chi} \in \mathcal{U}_R$ fixed, it is convenient to denote the solution to the linearized state system associated with a perturbation $\boldsymbol{h} \in \mathcal{U}$ as $(\xi^{\boldsymbol{h}}, \rho^{\boldsymbol{h}}, \boldsymbol{\omega}^{\boldsymbol{h}}, \zeta^{\boldsymbol{h}})$. From this result, it follows that the map

$$\mathcal{U} \subseteq L^2(Q) \times L^2(Q) \to \mathcal{V}, \qquad \boldsymbol{h} \mapsto (\xi^{\boldsymbol{h}}, \rho^{\boldsymbol{h}}, \boldsymbol{\omega}^{\boldsymbol{h}}, \zeta^{\boldsymbol{h}}),$$

is linear and continuous.

5.4 Differentiability of the control-to-state operator

In this section, we will prove the Fréchet differentiability of the control-to-state operator.

Theorem 5.22. *The control-to-state operator $\mathcal{S} : \mathcal{U} \to \mathcal{V}$ is Fréchet differentiable in $\mathcal{U}_R$ and the Fréchet derivative of $\mathcal{S}$ in $\boldsymbol{\chi} \in \mathcal{U}_R$ is given by*

$$D\mathcal{S}(\boldsymbol{\chi})\boldsymbol{h} = (\xi^{\boldsymbol{h}}, \rho^{\boldsymbol{h}}, \boldsymbol{\omega}^{\boldsymbol{h}}, \zeta^{\boldsymbol{h}})$$

for every $\boldsymbol{h} \in \mathcal{U}$.

Proof. We consider a fixed and arbitrary $\boldsymbol{\chi} \in \mathcal{U}_{\mathrm{ad}}$ with $\mathcal{S}(\boldsymbol{\chi}) = (\varphi, \sigma, \boldsymbol{u}, z)$. Our goal is to prove that

$$\lim_{\|\boldsymbol{h}\|_{\mathcal{U}} \to 0} \frac{\|\mathcal{S}(\boldsymbol{\chi} + \boldsymbol{h}) - \mathcal{S}(\boldsymbol{\chi}) - (\xi^{\boldsymbol{h}}, \rho^{\boldsymbol{h}}, \boldsymbol{\omega}^{\boldsymbol{h}}, \zeta^{\boldsymbol{h}})\|_{\mathcal{V}}}{\|\boldsymbol{h}\|_{\mathcal{U}}} = 0, \tag{5.4.1}$$

whence the thesis. We introduce the notation $\mathcal{S}(\boldsymbol{\chi} + \boldsymbol{h}) = (\varphi^{\boldsymbol{h}}, \sigma^{\boldsymbol{h}}, \boldsymbol{u}^{\boldsymbol{h}}, z^{\boldsymbol{h}})$ and

$$\Phi^{\boldsymbol{h}} := \varphi^{\boldsymbol{h}} - \varphi - \xi^{\boldsymbol{h}}, \quad \lambda^{\boldsymbol{h}} := \sigma^{\boldsymbol{h}} - \sigma - \rho^{\boldsymbol{h}}, \quad \boldsymbol{w}^{\boldsymbol{h}} := \boldsymbol{u}^{\boldsymbol{h}} - \boldsymbol{u} - \boldsymbol{\omega}^{\boldsymbol{h}}, \quad \mu^{\boldsymbol{h}} := z^{\boldsymbol{h}} - z - \zeta^{\boldsymbol{h}}.$$

Since $\boldsymbol{\chi}$ belongs to $\mathcal{U}_{\mathrm{ad}}$ which is in turn contained in the open set $\mathcal{U}_R$, there exists a constant $C_{\boldsymbol{\chi}}$ such that, for every $\boldsymbol{h} \in \mathcal{U}_R$ with $\|\boldsymbol{h}\|_{\mathcal{U}} \le C_{\boldsymbol{\chi}}$, the control $\boldsymbol{\chi} + \boldsymbol{h}$ still belongs to $\mathcal{U}_R$. Without loss of generality, since our aim is to pass to the limit as $\|\boldsymbol{h}\|_{\mathcal{U}}$ goes to 0, we will consider only $\boldsymbol{h}$ with a small norm in this sense. We are going to prove that

$$\|(\Phi^{\boldsymbol{h}}, \lambda^{\boldsymbol{h}}, \boldsymbol{w}^{\boldsymbol{h}}, \mu^{\boldsymbol{h}})\|_{\mathcal{V}} \le C_R \|\boldsymbol{h}\|_{\mathcal{U}}^{\frac{5}{4}}, \tag{5.4.2}$$

which yields the limit (5.4.1). To do so, we consider the PDE system satisfied by $(\Phi^{\boldsymbol{h}}, \lambda^{\boldsymbol{h}}, \boldsymbol{w}^{\boldsymbol{h}}, \mu^{\boldsymbol{h}})$ which can be trivially obtained by Theorem 5.5 and Proposition 5.20. Explicitly, the following equations are satisfied

$$\int_\Omega \partial_t \Phi^{\boldsymbol{h}} \eta \,\mathrm{d}x + \int_\Omega \nabla \Phi^{\boldsymbol{h}} \cdot \nabla \eta \,\mathrm{d}x = \int_\Omega [A_1 + A_2] \eta \,\mathrm{d}x, \tag{5.4.3a}$$

$$\langle \partial_t \lambda^{\boldsymbol{h}}, \eta \rangle_V + \int_\Omega \nabla \lambda^{\boldsymbol{h}} \cdot \nabla \eta \,\mathrm{d}x + \int_\Gamma \lambda^{\boldsymbol{h}} \eta \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega [B_1 + B_2 + B_3] \eta \,\mathrm{d}x, \tag{5.4.3b}$$

$$\int_\Omega \left[\mathcal{A}\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}}) + \mathfrak{C}_1 \right] : \varepsilon(\boldsymbol{\theta}) \,\mathrm{d}x = 0, \tag{5.4.3c}$$

$$\int_\Omega \partial_t \mu^{\boldsymbol{h}} \eta \,\mathrm{d}x + \nabla \mu^{\boldsymbol{h}} \cdot \nabla \eta \,\mathrm{d}x = \int_\Omega [D_1 + D_2] \,\eta \,\mathrm{d}x, \tag{5.4.3d}$$

for every $\eta \in V$ and $\boldsymbol{\theta} \in V_0$ as well as the initial conditions

$$\Phi^{\boldsymbol{h}}(0) = 0, \quad \lambda^{\boldsymbol{h}}(0) = 0, \quad \boldsymbol{w}^{\boldsymbol{h}}(0) = 0, \quad \mu^{\boldsymbol{h}}(0) = 0. \tag{5.4.4}$$

Here we have introduced the notation:

$$\begin{aligned} A_1 &= U(\varphi^{\boldsymbol{h}}, \sigma^{\boldsymbol{h}}, z^{\boldsymbol{h}}, \chi_1) - U(\varphi, \sigma, z, \chi_1) \\ &\qquad - \left[U_{,\varphi}(\varphi, \sigma, z, \chi_1)\xi^{\boldsymbol{h}} + U_{,\sigma}(\varphi, \sigma, z, \chi_1)\rho^{\boldsymbol{h}} + U_{,z}(\varphi, \sigma, z, \chi_1)\zeta^{\boldsymbol{h}} \right], \\ A_2 &= -\left[\varphi^{\boldsymbol{h}}\left(1 - \frac{\varphi^{\boldsymbol{h}}}{N}\right) - \varphi\left(1 - \frac{\varphi}{N}\right) \right] h_1, \end{aligned} \tag{5.4.5}$$

$$
\begin{aligned}
B_1 &= K(\varphi^{\boldsymbol{h}}, \sigma^{\boldsymbol{h}}, z^{\boldsymbol{h}}) - K(\varphi, \sigma, z) \\
&\quad - \left[K_{,\varphi}(\varphi, \sigma, z)\xi^{\boldsymbol{h}} + K_{,\sigma}(\varphi, \sigma, z)\rho^{\boldsymbol{h}} + K_{,z}(\varphi, \sigma, z)\zeta^{\boldsymbol{h}}\right], \\
B_2 &= \left[S(\varphi^{\boldsymbol{h}}, z^{\boldsymbol{h}}) - S(\varphi, z) - \left(S_{,\varphi}(\varphi, z)\xi^{\boldsymbol{h}} + S_{,z}(\varphi, z)\zeta^{\boldsymbol{h}}\right)\right]\chi_2, \\
B_3 &= [S(\varphi^{\boldsymbol{h}}, z^{\boldsymbol{h}}) - S(\varphi, z)]h_2, \\
\mathfrak{C}_1 &= \mathcal{B}(\varphi^{\boldsymbol{h}}, z^{\boldsymbol{h}})\varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{u}) \\
&\quad - \left[\mathcal{B}_{,\varphi}(\varphi, z)\varepsilon(\boldsymbol{u})\xi^{\boldsymbol{h}} + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{\omega}^{\boldsymbol{h}}) + \mathcal{B}_{,z}(\varphi, z)\varepsilon(\boldsymbol{u})\zeta^{\boldsymbol{h}}\right], \\
D_1 &= -\left[\beta(z^{\boldsymbol{h}}) + \pi(z^{\boldsymbol{h}}) - (\beta(z) + \pi(z)) - (\beta'(z) + \pi'(z))\zeta^{\boldsymbol{h}}\right], \\
D_2 &= -\left[F(\varphi^{\boldsymbol{h}}, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}})) - F(\varphi, \varepsilon(\boldsymbol{u})) - (F_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}))\xi^{\boldsymbol{h}} + F_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u})) : \varepsilon(\boldsymbol{\omega}^{\boldsymbol{h}}))\right].
\end{aligned} \tag{5.4.6}
$$

The next step is testing each equation in (5.4.3) with a proper term and doing some estimates. For this reason, it is convenient to rewrite some of the known coefficient functions we have just introduced. To this end, we recall that, according to Taylor's theorem with an integral remainder, for a function $l \in W^{2,2}([0,1])$ it holds

$$
l(1) = l(0) + l'(0) + \int_0^1 l''(s)(1-s)\,\mathrm{d}s.
$$

Let's take A_1 into account. We introduce $\boldsymbol{y}^{\boldsymbol{h}} = (\varphi^{\boldsymbol{h}}, \sigma^{\boldsymbol{h}}, z^{\boldsymbol{h}}, \chi_1)$, $\boldsymbol{y} = (\varphi, \sigma, z, \chi_1)$, and the function

$$
l(s) = U(s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y}),
$$

which is $W^{2,\infty}$ because U has this regularity and $s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y}$ is bounded. If we apply the formula above, we get

$$
\begin{aligned}
U(\boldsymbol{y}^{\boldsymbol{h}}) &= U(\boldsymbol{y}) + \nabla U(\boldsymbol{y}) \cdot (\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y}) \\
&\quad + \int_0^1 \left[D^2U(s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y})(\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y}) \cdot (\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y})\right](1-s)\,\mathrm{d}s \\
&=: U(\boldsymbol{y}) + \nabla U(\boldsymbol{y}) \cdot (\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y}) + \mathfrak{A}_1(\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y}) \cdot (\boldsymbol{y}^{\boldsymbol{h}} - \boldsymbol{y}).
\end{aligned}
$$

Notice that the matrix

$$
\mathfrak{A}_1 = \int_0^1 D^2U(s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y})(1-s)\,\mathrm{d}s
$$

as well as ∇U are bounded and their L^∞-norm are uniformly controlled by a constant that depends on R. Comparing the equality we have just obtained with A_1 leads to

$$
\begin{aligned}
A_1 &= U_{,\varphi}(\varphi, \sigma, z, \chi_1)\Phi^{\boldsymbol{h}} + U_{,\sigma}(\varphi, \sigma, z, \chi_1)\lambda^{\boldsymbol{h}} + U_{,z}(\varphi, \sigma, z, \chi_1)\mu^{\boldsymbol{h}} \\
&\quad + \mathfrak{A}_1(\varphi^{\boldsymbol{h}} - \varphi, \sigma^{\boldsymbol{h}} - \sigma, z^{\boldsymbol{h}} - z, 0) \cdot (\varphi^{\boldsymbol{h}} - \varphi, \sigma^{\boldsymbol{h}} - \sigma, z^{\boldsymbol{h}} - z, 0).
\end{aligned} \tag{5.4.7}
$$

Proceeding in the same way, we have:

$$
\begin{aligned}
B_1 &= K_{,\varphi}(\varphi, \sigma, z)\Phi^{\boldsymbol{h}} + K_{,\sigma}(\varphi, \sigma, z)\lambda^{\boldsymbol{h}} + K_{,z}(\varphi, \sigma, z)\mu^{\boldsymbol{h}} \\
&\quad + \mathfrak{B}_1(\varphi^{\boldsymbol{h}} - \varphi, \sigma^{\boldsymbol{h}} - \sigma, z^{\boldsymbol{h}} - z) \cdot (\varphi^{\boldsymbol{h}} - \varphi, \sigma^{\boldsymbol{h}} - \sigma, z^{\boldsymbol{h}} - z),
\end{aligned} \tag{5.4.8}
$$

$$B_2 = \Big[S_{,\varphi}(\varphi, z)\Phi^{\boldsymbol{h}} + S_{,z}(\varphi, z)\mu^{\boldsymbol{h}} + \mathfrak{B}_2(\varphi^{\boldsymbol{h}} - \varphi, z^{\boldsymbol{h}} - z) \cdot (\varphi^{\boldsymbol{h}} - \varphi, z^{\boldsymbol{h}} - z)\Big]\, \chi_2, \quad (5.4.9)$$

$$\begin{aligned}\mathfrak{C}_1 =& \mathcal{B}_{,\varphi}(\varphi, z)\varepsilon(\boldsymbol{u})\Phi^{\boldsymbol{h}} + \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{w}^{\boldsymbol{h}}) + \mathcal{B}_{,z}(\varphi, z)\varepsilon(\boldsymbol{u})\mu^{\boldsymbol{h}} \\ &+ \overline{\mathfrak{C}}_1(\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u}), z^{\boldsymbol{h}} - z) \cdot (\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u}), z^{\boldsymbol{h}} - z),\end{aligned} \quad (5.4.10)$$

$$D_1 = -\Big[(\beta'(z) + \pi'(z))\mu^{\boldsymbol{h}} + \mathfrak{D}_1(z^{\boldsymbol{h}} - z)^2\Big], \quad (5.4.11)$$

$$\begin{aligned}D_2 = -&\Big[F_{,\varphi}(\varphi, \varepsilon(\boldsymbol{u}))\Phi^{\boldsymbol{h}} + F_{,\varepsilon}(\varphi, \varepsilon(\boldsymbol{u})) : \varepsilon(\boldsymbol{w}^{\boldsymbol{h}}) \\ &+ \mathfrak{D}_2(\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u})) \cdot (\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u}))\Big].\end{aligned} \quad (5.4.12)$$

Notice that the terms written in Fraktur font are the ones related to the integral of the hessian of the auxiliary function l, and, therefore, their dimensions change from case to case: for example, $\mathfrak{B}_1$ is a matrix in $\mathbb{R}^{3\times3}$, $\mathfrak{B}_2$ a matrix in $\mathbb{R}^{2\times2}$, and $\mathfrak{D}_1$ is a scalar. Moreover, it is easy to check that all these terms are uniformly bounded in L^∞ by a constant that depends on R with the only exception of $\overline{\mathfrak{C}}_1$, which is uniformly bounded in $L^\infty(L^6(\Omega))$ because, even if $\mathcal{B} \in C^2(\mathbb{R}^2)$ with $\varphi, z \in L^\infty(Q)$, the terms $\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})$, $\varepsilon(\boldsymbol{u})$ are uniformly bounded only in this weaker norm. We will examine $\overline{\mathfrak{C}}_1$ more closely later, addressing the estimate of $\mathfrak{C}_1$. We test equation (5.4.3a) with $\Phi^{\boldsymbol{h}}$, obtaining

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\Phi^{\boldsymbol{h}}\|_H^2 = \int_\Omega (A_1 + A_2)\Phi^{\boldsymbol{h}}\,\mathrm{d}x.$$

From equation (5.4.7),

$$|A_1| \le C_R\left(|\Phi^{\boldsymbol{h}}| + |\lambda^{\boldsymbol{h}}| + |\mu^{\boldsymbol{h}}| + |\varphi^{\boldsymbol{h}} - \varphi|^2 + |\sigma^{\boldsymbol{h}} - \sigma|^2 + |z^{\boldsymbol{h}} - z|^2\right),$$

and from equation (5.4.5),

$$|A_2| \le C_R|\varphi^{\boldsymbol{h}} - \varphi|\,|h_1|.$$

Putting these elements together and using the Hölder inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\Phi^{\boldsymbol{h}}\|_H^2 \\ &\quad\le C_R\Big(\|\Phi^{\boldsymbol{h}}\|_H + \|\lambda^{\boldsymbol{h}}\|_H + \|\mu^{\boldsymbol{h}}\|_H\Big)\|\Phi^{\boldsymbol{h}}\|_H \\ &\qquad + C_R\Big(\|\varphi^{\boldsymbol{h}} - \varphi\|_H\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)} + \|\sigma^{\boldsymbol{h}} - \sigma\|_H\|\sigma^{\boldsymbol{h}} - \sigma\|_{L^6(\Omega)} \\ &\qquad + \|z^{\boldsymbol{h}} - z\|_H\|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)} + \|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}\|h_1\|_H\Big)\|\Phi^{\boldsymbol{h}}\|_{L^3(\Omega)}.\end{aligned}$$

We apply the Young inequality and then, we estimate the term $\|\Phi\|_{L^3(\Omega)}^2$ employing inequality (2.3.3). We have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\Phi^{\boldsymbol{h}}\|_H^2 &\le C_R\Big(\|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 + \|\varphi^{\boldsymbol{h}} - \varphi\|_H^2\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2 \\ &+ \|\sigma^{\boldsymbol{h}} - \sigma\|_H^2\|\sigma^{\boldsymbol{h}} - \sigma\|_{L^6(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_H^2\|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)}^2 \\ &+ \|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2\|h_1\|_H^2\Big) + C_{R,\delta}\|\Phi^{\boldsymbol{h}}\|_H^2 + \delta\|\nabla\Phi^{\boldsymbol{h}}\|_H^2\end{aligned} \quad (5.4.13)$$

for a small parameter $\delta > 0$. Testing (5.4.3b) with $\lambda^{\boldsymbol{h}}$ leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 \le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_{L^2_\Gamma}^2 = \int_\Omega (B_1 + B_2 + B_3)\lambda^{\boldsymbol{h}}\,\mathrm{d}x.$$

Proceeding as before, from equations (5.4.8) and (5.4.9) we get

$$|B_1| + |B_2| \le C_R\left(|\Phi^{\boldsymbol{h}}| + |\lambda^{\boldsymbol{h}}| + |\mu^{\boldsymbol{h}}| + |\varphi^{\boldsymbol{h}} - \varphi|^2 + |\sigma^{\boldsymbol{h}} - \sigma|^2 + |z^{\boldsymbol{h}} - z|^2\right),$$

where we have also employed the fact that $\|\chi_2\|_{L^\infty(Q)} \le R$. From equation (5.4.6) we derive

$$|B_3| \le C\left(|\varphi^{\boldsymbol{h}} - \varphi| + |z^{\boldsymbol{h}} - z|\right)|h_2|.$$

Consequently, through the Hölder and the Young inequalities, then the Gagliardo–Nirenberg inequality, and again the Young inequality with a small positive parameter δ, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 &\le C_R\Big(\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 + \|\varphi^{\boldsymbol{h}} - \varphi\|_H^2\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2 \\ &\quad + \|\sigma^{\boldsymbol{h}} - \sigma\|_H^2\|\sigma^{\boldsymbol{h}} - \sigma\|_{L^6(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_H^2\|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)}^2\Big) \\ &\quad + C_R\left(\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)}^2\right)\|h_2\|_H^2 + C_{R,\delta}\|\lambda^{\boldsymbol{h}}\|_H^2 + \delta\|\nabla\lambda^{\boldsymbol{h}}\|_H^2.\end{aligned} \tag{5.4.14}$$

We test equation (5.4.3c) with $\partial_t\boldsymbol{w}^{\boldsymbol{h}}$, obtaining:

$$C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}})\|_H^2 \le \int_\Omega \mathcal{A}\varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}}) : \varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}})\,\mathrm{d}x = -\int_\Omega \mathfrak{C}_1 : \varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}})\,\mathrm{d}x.$$

Our goal is to perform suitable estimates of the right-hand side of this inequality. By equation (5.4.10),

$$\begin{aligned}|\mathfrak{C}_1| &\le C|\varepsilon(\boldsymbol{u})|\left(|\Phi^{\boldsymbol{h}}| + |\mu^{\boldsymbol{h}}|\right) + C|\varepsilon(\boldsymbol{w}^{\boldsymbol{h}})| \\ &\quad + |\overline{\mathfrak{C}}_1(\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u}), z^{\boldsymbol{h}} - z)\cdot(\varphi^{\boldsymbol{h}} - \varphi, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) - \varepsilon(\boldsymbol{u}), z^{\boldsymbol{h}} - z)|.\end{aligned}$$

Let us analyze the last term on the right-hand side. We define $\boldsymbol{y}^{\boldsymbol{h}} = (\varphi^{\boldsymbol{h}}, \varepsilon(\boldsymbol{u}^{\boldsymbol{h}}), z^{\boldsymbol{h}})$, $\boldsymbol{y} = (\varphi, \varepsilon(\boldsymbol{u}), z)$, the function

$$L(\boldsymbol{y}) = \mathcal{B}(\varphi, z)\varepsilon(\boldsymbol{u}),$$

and the associated

$$l(s) = L\big(s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y}\big) = \mathcal{B}\big(s\varphi^{\boldsymbol{h}} + (1-s)\varphi, s\,z^{\boldsymbol{h}} + (1-s)\big)\left[s\,\varepsilon(\boldsymbol{u}^{\boldsymbol{h}}) + (1-s)\varepsilon(\boldsymbol{u})\right].$$

As done before,

$$\overline{\mathfrak{C}}_1 = \int_0^1 D^2L\big(s\boldsymbol{y}^{\boldsymbol{h}} + (1-s)\boldsymbol{y}\big)(1-s)\,\mathrm{d}s.$$

Notice that L is linear in $\varepsilon(\boldsymbol{u})$, so the related second derivative vanishes. Thus, the quadratic term in $\varepsilon(\boldsymbol{u^h}) - \varepsilon(\boldsymbol{u})$ will not appear in $\mathfrak{C}_1$. Explicitly,

$$\begin{aligned}|\overline{\mathfrak{C}}_1(\boldsymbol{y^h}-\boldsymbol{y})\cdot(\boldsymbol{y^h}-\boldsymbol{y})| \leq{}& C\left(|\varepsilon(\boldsymbol{u^h})|+|\varepsilon(\boldsymbol{u})|\right)\left[|\varphi^{\boldsymbol{h}}-\varphi|^2+|z^{\boldsymbol{h}}-z|^2\right]\\ &+C|\varepsilon(\boldsymbol{u^h})-\varepsilon(\boldsymbol{u})|\left[|\varphi^{\boldsymbol{h}}-\varphi|+|z^{\boldsymbol{h}}-z|\right],\end{aligned}$$

because $\mathcal{B}$ belongs to $C^2(\mathbb{R}^2)$ and its argument is uniformly bounded. Employing this inequality, we have

$$\begin{aligned}C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{w^h})\|_H^2 \leq{}& C\int_\Omega\left(|\varepsilon(\boldsymbol{u})|\left(|\Phi^{\boldsymbol{h}}|+|\mu^{\boldsymbol{h}}|\right)|\varepsilon(\partial_t\boldsymbol{w^h})|+|\varepsilon(\boldsymbol{w^h})||\varepsilon(\partial_t\boldsymbol{w^h})|\right)\mathrm{d}x\\ &+C\int_\Omega\left(|\varepsilon(\boldsymbol{u^h})-\varepsilon(\boldsymbol{u})|\right)\left(|\varphi^{\boldsymbol{h}}-\varphi|+|z^{\boldsymbol{h}}-z|\right)|\varepsilon(\partial_t\boldsymbol{w^h})|\,\mathrm{d}x\\ &+C\int_\Omega\left(|\varepsilon(\boldsymbol{u^h})|+|\varepsilon(\boldsymbol{u})|\right)\left(|\varphi^{\boldsymbol{h}}-\varphi|^2+|z^{\boldsymbol{h}}-z|^2\right)|\varepsilon(\partial_t\boldsymbol{w^h})|\,\mathrm{d}x\\ \leq{}& C\Big[\|\varepsilon(\boldsymbol{u})\|_{L^6(\Omega)}\left(\|\Phi^{\boldsymbol{h}}\|_{L^3(\Omega)}+\|\mu^{\boldsymbol{h}}\|_{L^3(\Omega)}\right)+\|\varepsilon(\boldsymbol{w^h})\|_H\\ &+\|\varepsilon(\boldsymbol{u^h})-\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)}\left(\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^4(\Omega)}+\|z^{\boldsymbol{h}}-z\|_{L^4(\Omega)}\right)\\ &+\left(\|\varepsilon(\boldsymbol{u^h})\|_{L^6(\Omega)}+\|\varepsilon(\boldsymbol{u})\|_{L^6(\Omega)}\right)\left(\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^2+\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^2\right)\Big]\|\varepsilon(\partial_t\boldsymbol{w^h})\|_H.\end{aligned}$$

We recall that by Theorem 5.7 the terms $\|\varepsilon(\boldsymbol{u})\|_{L^6(\Omega)}$, $\|\varepsilon(\boldsymbol{u^h})\|_{L^6(\Omega)}$ are bounded by a constant that depends on R. By the Young inequality and the Gagliardo–Nirenberg interpolation inequality from Lemma 2.6, we deduce

$$\begin{aligned}&C_{\mathcal{A}}\|\varepsilon(\partial_t\boldsymbol{w^h})\|_H^2\\ &\quad\leq\delta\left(\|\nabla\Phi^{\boldsymbol{h}}\|_H^2+\|\nabla\mu^{\boldsymbol{h}}\|_H^2+\|\varepsilon(\partial_t\boldsymbol{w^h})\|_H^2\right)+C_{R,\delta}\Big[\|\Phi^{\boldsymbol{h}}\|_H^2+\|\mu^{\boldsymbol{h}}\|_H^2\\ &\qquad+\|\varepsilon(\boldsymbol{w^h})\|_H^2+\|\varepsilon(\boldsymbol{u^h})-\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)}^2(\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^4(\Omega)}^2+\|z^{\boldsymbol{h}}-z\|_{L^4(\Omega)}^2)\\ &\qquad+\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^4+\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^4\Big]\end{aligned}\tag{5.4.15}$$

for a small $\delta>0$. Testing equation (5.4.3d) with $\mu^{\boldsymbol{h}}$ leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mu^{\boldsymbol{h}}\|_H^2+\|\nabla\mu^{\boldsymbol{h}}\|_H^2=\int_\Omega(D_1+D_2)\mu^{\boldsymbol{h}}\,\mathrm{d}x.$$

Regarding D_1, we recall that thanks to the separation property we proved for z, the term $\beta'(z)+\pi'(z)$ is bounded. We have

$$|D_1|\leq C\left(|\mu^{\boldsymbol{h}}|+|z^{\boldsymbol{h}}-z|^2\right).$$

Turning our attention to D_2, we get

$$|D_2|\leq C\left(|\Phi^{\boldsymbol{h}}|+|\varepsilon(\boldsymbol{w^h})|+|\varphi^{\boldsymbol{h}}-\varphi|^2+|\varepsilon(\boldsymbol{u^h})-\varepsilon(\boldsymbol{u})|^2\right).$$

Thus, with standard argumentation, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mu^{\boldsymbol{h}}\|_H^2+\|\nabla\mu^{\boldsymbol{h}}\|_H^2\\&\quad\le\delta\|\nabla\mu^{\boldsymbol{h}}\|_H^2+C_\delta\Big(\|\mu^{\boldsymbol{h}}\|_H^2+\|\Phi^{\boldsymbol{h}}\|_H^2+\|\varepsilon(\boldsymbol{w}^{\boldsymbol{h}})\|_H^2\\&\quad+\|z^{\boldsymbol{h}}-z\|_H^2\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^2+\|\varphi^{\boldsymbol{h}}-\varphi\|_H^2\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^2\\&\quad+\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_H^2\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)}^2\Big).\end{aligned}\tag{5.4.16}$$

We sum inequalities (5.4.13)–(5.4.16), integrate in time over $(0,t)$ remembering that the initial values of the variables are zero, and move the terms multiplied by δ to the left-hand side, fixing a parameter small enough. This way, we get the following:

$$\begin{aligned}&\|\Phi^{\boldsymbol{h}}\|_H^2+\|\lambda^{\boldsymbol{h}}\|_H^2+\|\mu^{\boldsymbol{h}}\|_H^2+\int_0^t\Big(\|\nabla\Phi^{\boldsymbol{h}}\|_H^2+\|\nabla\lambda^{\boldsymbol{h}}\|_H^2+\|\varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}})\|_H^2+\|\nabla\mu^{\boldsymbol{h}}\|_H^2\Big)\,\mathrm{d}s\\&\quad\le C_R\int_0^t\Big[\|\Phi^{\boldsymbol{h}}\|_H^2+\|\lambda^{\boldsymbol{h}}\|_H^2+\|\varepsilon(\boldsymbol{w}^{\boldsymbol{h}})\|_H^2+\|\mu^{\boldsymbol{h}}\|_H^2\\&\qquad+\|\varphi^{\boldsymbol{h}}-\varphi\|_H^2\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^2+\|\sigma^{\boldsymbol{h}}-\sigma\|_H^2\|\sigma^{\boldsymbol{h}}-\sigma\|_{L^6(\Omega)}^2+\|z^{\boldsymbol{h}}-z\|_H^2\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^2\\&\qquad+\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^2\|h_1\|_H^2+\Big(\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^2+\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^2\Big)\|h_2\|_H^2\\&\qquad+\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)}^2(\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^4(\Omega)}^2+\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_H^2+\|z^{\boldsymbol{h}}-z\|_{L^4(\Omega)}^2)\\&\qquad+\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^4+\|z^{\boldsymbol{h}}-z\|_{L^6(\Omega)}^4\Big]\,\mathrm{d}s.\end{aligned}$$

We recall that

$$\|\varepsilon(\boldsymbol{w}^{\boldsymbol{h}})\|_H^2\le C\int_0^s\|\varepsilon(\partial_t\boldsymbol{w}^{\boldsymbol{h}})\|_H^2\,\mathrm{d}\tau,$$

and that, thanks to Theorem 5.8, it holds

$$\begin{aligned}&\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^\infty(H)}+\|\sigma^{\boldsymbol{h}}-\sigma\|_{L^\infty(H)}\\&\quad+\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_{L^\infty(H)}+\|z^{\boldsymbol{h}}-z\|_{L^\infty(H)}\le C_R\|\boldsymbol{h}\|_{L^2(Q)},\end{aligned}$$

and that, by (5.1.94) in Remark 5.13, we have

$$\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^\infty(L^4(\Omega))}+\|\varepsilon(\boldsymbol{u}^{\boldsymbol{h}})-\varepsilon(\boldsymbol{u})\|_{L^\infty(L^4(\Omega))}+\|z^{\boldsymbol{h}}-z\|_{L^\infty(L^4(\Omega))}\le C_R\|\boldsymbol{h}\|_{L^2(Q)}^{\frac14}.$$

Finally, we observe that

$$\begin{aligned}&\int_0^t\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^6(\Omega)}^4\,\mathrm{d}s\le\int_0^t\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^\infty(\Omega)}^2\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^3(\Omega)}^2\,\mathrm{d}s\\&\quad\le\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^4(L^\infty(\Omega))}^2\|\varphi^{\boldsymbol{h}}-\varphi\|_{L^4(L^3(\Omega))}^2\le C_R\|\boldsymbol{h}\|_{L^2(Q)}^{\frac12}\|\boldsymbol{h}\|_{L^2(Q)}^2=C_R\|\boldsymbol{h}\|_{L^2(Q)}^{\frac52}\end{aligned}$$

where we have applied the Hölder inequality and, in the last passage, we have combined (5.1.95) and (5.1.96) from Remark 5.13. The same inequality holds for $z^{\boldsymbol{h}}-z$. Thus, we obtain

$$\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 + \int_0^t \Big(\|\nabla\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 + \|\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}})\|_H^2 + \|\nabla\mu^{\boldsymbol{h}}\|_H^2 \Big) \,\mathrm{d}s$$
$$\leq C_R \bigg\{ \int_0^t \Big[\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 \Big] \,\mathrm{d}s + \int_0^t \int_0^s \|\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}})\|_H^2 \,\mathrm{d}\tau \,\mathrm{d}s$$
$$+ \|\boldsymbol{h}\|_{L^2(Q)}^2 \int_0^t \Big[\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2 + \|\sigma^{\boldsymbol{h}} - \sigma\|_{L^6(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)}^2 \Big] \,\mathrm{d}s$$
$$+ \|\boldsymbol{h}\|_{L^\infty(H)}^2 \int_0^t \Big[\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^6(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_{L^6(\Omega)}^2 \Big] \,\mathrm{d}s$$
$$+ \|\boldsymbol{h}\|_{L^2(Q)}^{\frac{1}{2}} \int_0^t \Big[\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^4(\Omega)}^2 + \|z^{\boldsymbol{h}} - z\|_{L^4(\Omega)}^2 \Big] \,\mathrm{d}s + \|\boldsymbol{h}\|_{L^2(Q)}^4 + \|\boldsymbol{h}\|_{L^2(Q)}^{\frac{5}{2}} \bigg\}.$$

Again, we recall that from Theorem 5.8 we know

$$\|\varphi^{\boldsymbol{h}} - \varphi\|_{L^2(V)} + \|\sigma^{\boldsymbol{h}} - \sigma\|_{L^2(V)} + \|z^{\boldsymbol{h}} - z\|_{L^2(V)} \leq C_R \|\boldsymbol{h}\|_{L^2(Q)},$$

and that $V \hookrightarrow L^4(\Omega), L^6(\Omega)$. Finally, we obtain:

$$\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 + \int_0^t \Big(\|\nabla\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 + \|\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}})\|_H^2 + \|\nabla\mu^{\boldsymbol{h}}\|_H^2 \Big) \,\mathrm{d}s$$
$$\leq C_R \bigg\{ \int_0^t \Big[\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 \Big] \,\mathrm{d}s + \int_0^t \int_0^s \|\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}})\|_H^2 \,\mathrm{d}\tau \,\mathrm{d}s$$
$$+ \|\boldsymbol{h}\|_{L^2(Q)}^4 + \|\boldsymbol{h}\|_{L^\infty(H)}^2 \|\boldsymbol{h}\|_{L^2(Q)}^2 + \|\boldsymbol{h}\|_{L^2(Q)}^{\frac{5}{2}} \bigg\}.$$

By means of the Gronwall inequality, we have:

$$\|\Phi^{\boldsymbol{h}}\|_H^2 + \|\lambda^{\boldsymbol{h}}\|_H^2 + \|\mu^{\boldsymbol{h}}\|_H^2 + \int_0^t \Big(\|\nabla\Phi^{\boldsymbol{h}}\|_H^2 + \|\nabla\lambda^{\boldsymbol{h}}\|_H^2 + \|\varepsilon(\partial_t \boldsymbol{w}^{\boldsymbol{h}})\|_H^2 + \|\nabla\mu^{\boldsymbol{h}}\|_H^2 \Big) \,\mathrm{d}s$$
$$\leq C_R \Big(\|\boldsymbol{h}\|_{L^2(Q)}^4 + \|\boldsymbol{h}\|_{L^\infty(H)}^2 \|\boldsymbol{h}\|_{L^2(Q)}^2 + \|\boldsymbol{h}\|_{L^2(Q)}^{\frac{5}{2}} \Big),$$

whence (5.4.2) follows. Therefore, the proof of Theorem 5.22 is complete. □

From the Fréchet differentiability of $\mathcal{S}$, it follows that the reduced cost functional J is Fréchet differentiable over the set $\mathcal{U}_R$. Since $\mathcal{U}_{\text{ad}}$ is a closed and convex subset of $\mathcal{U}$, we can prove the following result.

Corollary 5.23. *Let $\boldsymbol{\chi}^* \in \mathcal{U}_{ad}$ be an optimal control for the control problem with the* *associated state* $\mathcal{S}(\boldsymbol{\chi}^*) = (\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*)$*. Then, the following inequality is satisfied*

$$
\begin{aligned}
&\alpha_1 \int_0^T\!\!\int_\Omega (\varphi^* - \varphi_Q)\xi \,\mathrm{d}x\,\mathrm{d}t + \alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T)\,\mathrm{d}x + \alpha_3 \int_\Omega \xi(T)\,\mathrm{d}x \\
&\quad + \alpha_4 \int_0^T\!\!\int_\Omega (\sigma^* - \sigma_Q)\rho \,\mathrm{d}x\,\mathrm{d}t + \alpha_5 \int_\Omega (\sigma^*(T) - \sigma_\Omega)\rho(T)\,\mathrm{d}x \\
&\quad + \alpha_6 \int_0^T\!\!\int_\Omega \left[\frac{1}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*)\xi + \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{\omega})\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha_7 \int_0^T\!\!\int_\Omega (z^* - z_Q)\zeta \,\mathrm{d}x\,\mathrm{d}t + \alpha_8 \int_\Omega \zeta(T)\,\mathrm{d}x \\
&\quad + \alpha_9 \int_0^T\!\!\int_\Omega \boldsymbol{\chi}^* \cdot (\boldsymbol{\chi} - \boldsymbol{\chi}^*)\,\mathrm{d}x\,\mathrm{d}t \geq 0
\end{aligned}
\tag{5.4.17}
$$

for every $\boldsymbol{\chi} \in \mathcal{U}_{ad}$*, where* $(\xi, \rho, \boldsymbol{\omega}, \zeta)$ *is the unique solution to the linearized system in* $\boldsymbol{\chi}^*$ *for* $\boldsymbol{h} = \boldsymbol{\chi} - \boldsymbol{\chi}^*$*.*

Proof. First of all, we recall that the cost functional $\mathcal{J}$ is well-defined over the space

$$
\Big[C^0([0,T];H) \times C^0([0,T];H) \times L^2(0,T;V) \times C^0([0,T];H)\Big] \times \Big[L^2(0,T;H) \times L^2(0,T;H)\Big],
$$

where it is also Fréchet differentiable. Moreover, from Theorem 5.22, the solution operator $\mathcal{S} : \mathcal{U} \to \mathcal{V}$ is Fréchet differentiable in $\mathcal{U}_R$. Since, from standard embedding results, $\mathcal{V}$ is continuously embedded in

$$
C^0([0,T];H) \times C^0([0,T];H) \times L^2(0,T;V) \times C^0([0,T];H),
$$

$\mathcal{S}$ is as well Fréchet differentiable in $\mathcal{U}_R$ if it is seen as a functional between $\mathcal{U}$ and this larger space. Thus, the reduced cost functional J is Fréchet differentiable in $\mathcal{U}_R$ and, by the chain rule,

$$
\begin{aligned}
&D\mathrm{J}(\boldsymbol{\chi}^*)[\boldsymbol{\chi} - \boldsymbol{\chi}^*] \\
&\quad = \alpha_1 \int_0^T\!\!\int_\Omega (\varphi^* - \varphi_Q)\xi \,\mathrm{d}x\,\mathrm{d}t + \alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T)\,\mathrm{d}x \\
&\quad + \alpha_3 \int_\Omega \xi(T)\,\mathrm{d}x + \alpha_4 \int_0^T\!\!\int_\Omega (\sigma^* - \sigma_Q)\rho \,\mathrm{d}x\,\mathrm{d}t + \alpha_5 \int_\Omega (\sigma^*(T) - \sigma_\Omega)\rho(T)\,\mathrm{d}x \\
&\quad + \alpha_6 \int_0^T\!\!\int_\Omega \left[\frac{1}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*)\xi + \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{\omega})\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha_7 \int_0^T\!\!\int_\Omega (z^* - z_Q)\zeta \,\mathrm{d}x\,\mathrm{d}t + \alpha_8 \int_\Omega \zeta(T)\,\mathrm{d}x + \alpha_9 \int_0^T\!\!\int_\Omega \boldsymbol{\chi}^* \cdot (\boldsymbol{\chi} - \boldsymbol{\chi}^*)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where $\boldsymbol{\chi}^*$ a the optimal control, $\boldsymbol{\chi}$ is any admissible control, and $(\xi, \rho, \boldsymbol{\omega}, \zeta)$ is the solution to the system linearized in $\boldsymbol{\chi}^*$ with $\boldsymbol{h} = \boldsymbol{\chi} - \boldsymbol{\chi}^*$. From the optimality of $\boldsymbol{\chi}^*$ and the convexity of $\mathcal{U}_{\mathrm{ad}}$, we obtain the trivial inequality

$$
\mathrm{J}(\boldsymbol{\chi}^* + t(\boldsymbol{\chi} - \boldsymbol{\chi}^*)) - \mathrm{J}(\boldsymbol{\chi}^*) \geq 0
$$

for every $t \in (0,1)$. Dividing by t and passing to the limit as $t \to 0^+$ leads to

$$DJ(\boldsymbol{\chi}^*)[\boldsymbol{\chi} - \boldsymbol{\chi}^*] \geq 0.$$

□

The next step is simplifying the expression (5.4.17), removing the linearized variables. In fact, even though the inequality just derived represents a first-order necessary condition for optimality, it does not provide a practically efficient characterization. Specifically, for every element $\boldsymbol{\chi}$ of the admissible control space $\mathcal{U}_{\mathrm{ad}}$, it requires solving the corresponding linearized system for $\boldsymbol{h} = \boldsymbol{\chi} - \boldsymbol{\chi}^*$. This approach is computationally demanding, particularly in high-dimensional control spaces, highlighting the need for a reformulation. To this end, we need to introduce the adjoint system.

5.5 The adjoint system and first-order necessary optimality conditions

The adjoint system associated with an optimal control $\boldsymbol{\chi}^* \in \mathcal{U}_{\mathrm{ad}}$ and its corresponding solution to the state system $(\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*) = \mathcal{S}(\boldsymbol{\chi}^*)$ is given by

$$\begin{aligned} -\partial_t \mathrm{q} - \Delta \mathrm{q} &= a_1 \mathrm{q} + b_1 \mathrm{r} + d_1 \mathrm{s} + \mathfrak{c}_1 : \varepsilon(\mathbf{v}) \\ &\quad + \alpha_1(\varphi^* - \varphi_Q) + \frac{\alpha_6}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*), \end{aligned} \tag{5.5.1a}$$

$$-\partial_t \mathrm{r} - \Delta \mathrm{r} = a_2 \mathrm{q} + b_2 \mathrm{r} + \alpha_4(\sigma^* - \sigma_Q), \tag{5.5.1b}$$

$$-\operatorname{div}\left[-\mathcal{A}\varepsilon(\partial_t \mathbf{v}) + \mathcal{B}(\varphi^*, z^*)\varepsilon(\mathbf{v})\right] = -\operatorname{div}\left[\mathfrak{d}_2 \mathrm{s} + \alpha_6 \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*)\right], \tag{5.5.1c}$$

$$-\partial_t \mathrm{s} - \Delta \mathrm{s} = a_3 \mathrm{q} + b_3 \mathrm{r} + d_3 \mathrm{s} + \mathfrak{c}_2 : \varepsilon(\mathbf{v}) + \alpha_7(z^* - z_Q), \tag{5.5.1d}$$

coupled with the boundary conditions

$$\partial_{\boldsymbol{\nu}} \mathrm{q} = 0, \quad \partial_{\boldsymbol{\nu}} \mathrm{r} = -\mathrm{r}, \quad \mathbf{v} = 0, \quad \partial_{\boldsymbol{\nu}} \mathrm{s} = 0, \tag{5.5.2}$$

and with the final conditions

$$\mathrm{q}(T) = \alpha_2(\varphi^*(T) - \varphi_\Omega) + \alpha_3, \quad \mathrm{r}(T) = \alpha_5(\sigma^*(T) - \sigma_\Omega), \quad \mathbf{v}(T) = 0, \quad \mathrm{s}(T) = \alpha_8. \tag{5.5.3}$$

Proposition 5.24. *Let $\boldsymbol{\chi}^*$ be an optimal control with the associated solution to the state system $(\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*) = \mathcal{S}(\boldsymbol{\chi}^*)$. The adjoint system (5.5.1)–(5.5.3) has a unique weak solution $(\mathrm{q}, \mathrm{r}, \mathbf{v}, \mathrm{s})$ in the sense that*

$$\begin{gathered} \mathrm{q} \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \\ \mathrm{r} \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \\ \mathbf{v} \in W^{1,\infty}(0,T;V_0), \\ \mathrm{s} \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \end{gathered}$$

with

$$\mathrm{q}(T) = \alpha_2(\varphi^*(T) - \varphi_\Omega) + \alpha_3, \quad \mathrm{r}(T) = \alpha_5(\sigma^*(T) - \sigma_\Omega), \quad \mathbf{v}(T) = 0, \quad \mathrm{s}(T) = \alpha_8,$$

such that

$$\begin{aligned} -\langle \partial_t \mathrm{q}, \eta \rangle_V + \int_\Omega \nabla \mathrm{q} \cdot \nabla \eta \,\mathrm{d}x &= \int_\Omega [a_1 \mathrm{q} + b_1 \mathrm{r} + d_1 \mathrm{s} + \mathfrak{c}_1 : \varepsilon(\mathbf{v})]\, \eta \,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\alpha_1(\varphi^* - \varphi_Q) + \frac{\alpha_6}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*)\Big]\, \eta \,\mathrm{d}x, \end{aligned} \tag{5.5.4a}$$

$$-\langle \partial_t \mathrm{r}, \eta \rangle_V + \int_\Omega \nabla \mathrm{r} \cdot \nabla \eta \,\mathrm{d}x = -\int_\Gamma \mathrm{r}\, \eta \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega [a_2 \mathrm{q} + b_2 \mathrm{r} + \alpha_4(\sigma^* - \sigma_Q)]\, \eta \,\mathrm{d}x, \tag{5.5.4b}$$

$$\int_\Omega [-\mathcal{A}\varepsilon(\partial_t \mathbf{v}) + \mathcal{B}(\varphi^*, z^*)\varepsilon(\mathbf{v})] : \varepsilon(\boldsymbol{\theta}) \,\mathrm{d}x = \int_\Omega [\mathfrak{d}_2 \mathrm{s} + \alpha_6 \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*)] : \varepsilon(\boldsymbol{\theta}) \,\mathrm{d}x, \tag{5.5.4c}$$

$$\int_\Omega (-\partial_t \mathrm{s} \eta + \nabla \mathrm{s} \cdot \nabla \eta) \,\mathrm{d}x = \int_\Omega [a_3 \mathrm{q} + b_3 \mathrm{r} + d_3 \mathrm{s} + \mathfrak{c}_2 : \varepsilon(\mathbf{v}) + \alpha_7(z^* - z_Q)]\, \eta \,\mathrm{d}x, \tag{5.5.4d}$$

a.e. in $(0, T)$*, for every* $\eta \in V$ *and* $\boldsymbol{\theta} \in V_0$*.*

Notice that the regularity of q is the maximal we can expect, since the final value $\alpha_2(\varphi^*(T) - \varphi_\Omega) + \alpha_3$ only belongs to H.

Proof. The proof is similar to that of Proposition 5.20. Thus, we will omit the full details. Existence can be established via a standard Galerkin scheme. For this reason, we present only the formal a priori estimates required to justify the passage from the discrete to the continuous system. We test equation (5.5.4a) with q, equation (5.5.4b) with r, and equation (5.5.4d) with s. We sum all these equalities and estimate the right-hand side with the Hölder and Young inequalities, obtaining:

$$\begin{aligned} &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\mathrm{q}|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{r}|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{s}|^2 \,\mathrm{d}x\right) \\ &\quad + \int_\Omega |\nabla \mathrm{q}|^2 \,\mathrm{d}x + \int_\Omega |\nabla \mathrm{r}|^2 \,\mathrm{d}x + \int_\Omega |\nabla \mathrm{s}|^2 \,\mathrm{d}x \\ &\le C_R \left(\int_\Omega |\mathrm{q}|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{r}|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{s}|^2 \,\mathrm{d}x + \int_\Omega |\varepsilon(\mathbf{v})|^2 \,\mathrm{d}x + 1\right) \\ &\quad + C_R \left(\|\mathrm{q}\|^2_{L^3(\Omega)} + \|\mathrm{s}\|^2_{L^3(\Omega)}\right), \end{aligned}$$

where we employed Remark 5.19 to bound the coefficients of the linear terms, and Theorem 5.7 for the optimal state. We deal with the L^3-norms by means of inequality (2.3.3), moving to the left-hand side the resulting gradient terms. Integrating in time over the time interval (t,T), we have

$$\begin{aligned}&\int_\Omega |\mathrm{q}|^2\,\mathrm{d}x + \int_\Omega |\mathrm{r}|^2\,\mathrm{d}x + \int_\Omega |\mathrm{s}|^2\,\mathrm{d}x \\ &\qquad + \int_t^T\!\!\int_\Omega |\nabla \mathrm{q}|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Omega |\nabla \mathrm{r}|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Omega |\nabla \mathrm{s}|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\le C_T + C_R\Big(\int_t^T\!\!\int_\Omega |\mathrm{q}|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Omega |\mathrm{r}|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_t^T\!\!\int_\Omega |\mathrm{s}|^2\,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Omega |\varepsilon(\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}s + 1\Big),\end{aligned} \tag{5.5.5}$$

where the non-negative constant C_T only depends on the prescribed final conditions (5.5.3). Then, we test equation (5.5.4c) with $-\partial_t\mathbf{v}$. We use the fact that $\mathcal{A}$ is strictly positive definite from hypothesis (C2) to bound from below the left-hand side, and again the Hölder and Young inequalities for the right-hand side. We end up with

$$\begin{aligned}&C_{\mathcal{A}}\int_\Omega |\varepsilon(\partial_t\mathbf{v})|^2\,\mathrm{d}x \le \int_\Omega \mathcal{A}\varepsilon(\partial_t\mathbf{v}) : \varepsilon(\partial_t\mathbf{v})\,\mathrm{d}x \\ &\quad \le \delta\int_\Omega |\varepsilon(\partial_t\mathbf{v})|^2\,\mathrm{d}x + C_{R,\delta}\left(\int_\Omega |\varepsilon(\mathbf{v})|^2\,\mathrm{d}x + \int_\Omega |s|^2\,\mathrm{d}x + 1\right)\end{aligned} \tag{5.5.6}$$

for a fixed and small constant $\delta > 0$, e.g., $\delta = C_{\mathcal{A}}/2$. We recall that, since $\mathbf{v}(T) = 0$, as we have already seen before,

$$\int_\Omega |\varepsilon(\mathbf{v})|^2\,\mathrm{d}x \le \int_t^T\!\!\int_\Omega |\varepsilon(\partial_t\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}s$$

that we are going to use in equation (5.5.6). Similarly, integrating this inequality in time,

$$\int_t^T\!\!\int_\Omega |\varepsilon(\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_t^T \left(\int_s^T\!\!\int_\Omega |\varepsilon(\partial_t\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}\tau\right)\,\mathrm{d}s \le T\int_t^T\!\!\int_\Omega |\varepsilon(\partial_t\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}s,$$

that we are going to use in equation (5.5.5). Summing inequalities (5.5.5) and (5.5.6) and employing the Gronwall Lemma backward in time, leads to

$$\|\mathrm{q}\|_{L^\infty(H)\cap L^2(V)} + \|\mathrm{r}\|_{L^\infty(H)\cap L^2(V)} + \|\mathrm{s}\|_{L^\infty(H)\cap L^2(V)} + \|\varepsilon(\partial_t\mathbf{v})\|_{L^\infty(V_0)} \le C. \tag{5.5.7}$$

Additional estimates in $H^1(0,T;V')$ for q, r and s can be deduced by comparison in the respective equations. Moreover, since the final value of s is smooth, we can apply standard parabolic results (see, e.g., [DL92; Lio61]) and improve its regularity. Uniqueness is easy to prove since the system is linear. One can take the difference of two weak solutions, which is in turn a solution of the homogeneous system associated with (5.5.1) with zero final conditions, and repeat the estimate above. □

We are now in the position to remove the linearized variables from (5.4.17).

Theorem 5.25 (First-order necessary conditions of optimality)**.** *Let $\boldsymbol{\chi}^* \in \mathcal{U}_{ad}$ be an optimal control for the control problem with the associated state $(\varphi^*, \sigma^*, \boldsymbol{u}^*, z^*) = \mathcal{S}(\boldsymbol{\chi}^*)$. Then, the following inequality is satisfied*

$$
\begin{aligned}
&- \int_0^T\!\!\int_\Omega \varphi^* \Big(1 - \frac{\varphi^*}{N}\Big)(\chi_1 - \chi_1^*)\mathrm{q} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega S(\varphi^*, z^*)(\chi_2 - \chi_2^*)\mathrm{r}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \alpha_9 \int_0^T\!\!\int_\Omega \boldsymbol{\chi}^* \cdot (\boldsymbol{\chi} - \boldsymbol{\chi}^*)\,\mathrm{d}x\,\mathrm{d}t \geq 0
\end{aligned}
\tag{5.5.8}
$$

for every $\boldsymbol{\chi} \in \mathcal{U}_{ad}$, where q *and* r *are respectively the first and second components of the solution to the adjoint system associated with $\boldsymbol{\chi}^*$.*

Proof. We test the equations of the linearized system with the solution of the adjoint system and subtract the equations of the adjoint system tested with the solution of the linearized system. Then, we sum up all the equations we have obtained. Explicitly, we test (5.3.4a) with q, (5.3.4b) with r, (5.3.4c) with $\mathbf{v}$ and (5.3.4d) with s. In the same way, we test (5.5.4a) with ξ, (5.5.4b) with ρ, (5.5.4c) with $\boldsymbol{\omega}$ and (5.5.4d) with ζ. Some terms cancel out, and we have:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega (\xi\mathrm{q} + \rho\mathrm{r} + \mathcal{A}\varepsilon(\boldsymbol{\omega}) : \varepsilon(\mathbf{v}) + \zeta\mathrm{s})\ \mathrm{d}x\right] \\
&\quad = \int_\Omega (a_4 h_1 \mathrm{q} + b_4 h_2 \mathrm{r})\ \mathrm{d}x - \alpha_1 \int_\Omega (\varphi^* - \varphi_Q)\xi\,\mathrm{d}x - \alpha_4 \int_\Omega (\sigma^* - \sigma_Q)\rho\,\mathrm{d}x \\
&\qquad - \alpha_6 \int_\Omega \Big(\frac{1}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*)\xi + \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{\omega})\Big)\ \mathrm{d}x - \alpha_7 \int_\Omega (z^* - z_Q)\zeta\,\mathrm{d}x.
\end{aligned}
$$

Integrating in time over the interval $(0, T)$, exploiting the initial and final conditions, and moving some terms to the left-hand side, we finally have

$$
\begin{aligned}
&\Big[\alpha_2 \int_\Omega (\varphi^*(T) - \varphi_\Omega)\xi(T)\,\mathrm{d}x + \alpha_3 \int_\Omega \xi(T)\,\mathrm{d}x + \alpha_5 \int_\Omega (\sigma^*(T) - \sigma_\Omega)\rho(T)\,\mathrm{d}x \\
&\quad + \alpha_8 \int_\Omega \zeta(T)\,\mathrm{d}x\Big] + \Big[\alpha_1 \int_0^T\!\!\int_\Omega (\varphi^* - \varphi_Q)\xi\,\mathrm{d}x\,\mathrm{d}t + \alpha_4 \int_0^T\!\!\int_\Omega (\sigma^* - \sigma_Q)\rho\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha_6 \int_0^T\!\!\int_\Omega \Big(\frac{1}{2}\gamma'(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{u}^*)\xi + \gamma(\varphi^*)\varepsilon(\boldsymbol{u}^*) : \varepsilon(\boldsymbol{\omega})\Big)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \alpha_7 \int_0^T\!\!\int_\Omega (z^* - z_Q)\zeta\,\mathrm{d}x\,\mathrm{d}t\Big] = \int_0^T\!\!\int_\Omega (a_4 h_1 \mathrm{q} + b_4 h_2 \mathrm{r})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Combining this equality with the inequality stated in Corollary 5.23, and recalling the expressions of a_4 and b_4 from the linearized system, we obtain the thesis. ☐

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